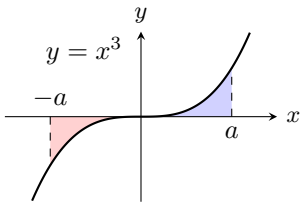
Definitions and integrals

An **odd** function $f(x)$ has

$$f(-x) = -f(x) \quad \text{for all real } x$$

Antisymmetry in the y-axis ($x = 0$): half-turn symmetry about O.



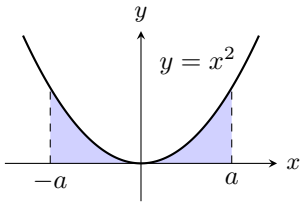
$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 0$$

Examples: x, x^3, $x^5 - 2x$, $\sin x$

An **even** function $f(x)$ has

$$f(-x) = f(x) \quad \text{for all real } x$$

Symmetry in the y-axis ($x = 0$): the y-axis is a line of symmetry.



$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_{0}^{a} f(x)\,\mathrm{d}x$$

Examples: 1, x^2, $|x|$, $\cos x$

Theorem

Every function $f(x)$ defined for all real x is **uniquely** the sum of an odd function and an even function:

$$f(x) = f_{\text{odd}}(x) + f_{\text{even}}(x) \qquad \text{where} \qquad f_{\text{odd}}(x) = \frac{f(x) - f(-x)}{2} \qquad f_{\text{even}}(x) = \frac{f(x) + f(-x)}{2}$$

A polynomial splits into its odd powers and its even powers, and e^x, which is neither odd nor even, splits into $\sinh x$ and $\cosh x$:

$$x^3 + 4x^2 - 2x + 7 = \underbrace{(x^3 - 2x)}_{\text{odd}} + \underbrace{(4x^2 + 7)}_{\text{even}} \qquad e^x = \underbrace{\frac{e^x - e^{-x}}{2}}_{\sinh x,\ \text{odd}} + \underbrace{\frac{e^x + e^{-x}}{2}}_{\cosh x,\ \text{even}}$$

Over an interval from $-a$ to a only the even part survives:

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_{0}^{a} f_{\text{even}}(x)\,\mathrm{d}x$$

Products and compositions

Products (and quotients, where defined)

odd × even = odd	$x^3 \times x^2 = x^5$
odd × odd = even	$x \sin x$
even × even = even	$x^2 \cos x$

Sums: odd + odd = odd, even + even = even

Compositions $f(g(x))$

g odd and f odd: odd	$\sin(x^3)$
g odd and f even: even	$\cos(x^3)$
g even: even, whatever f is	e^{-x^2}, $f(\lvert x\rvert)$

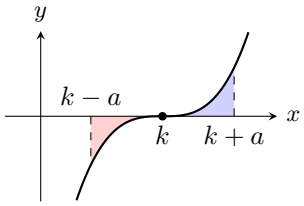
$f(|x|)$ is the even function equal to $f(x)$ for $x \geq 0$

Antisymmetry and symmetry in the line $x = k$

$f(x)$ is **antisymmetric in** $x = k$ if

$$f(k + x) = -f(k - x) \quad \text{for all real } x$$

that is, if $g(x) = f(x + k)$ is odd: half-turn symmetry about $(k, 0)$.



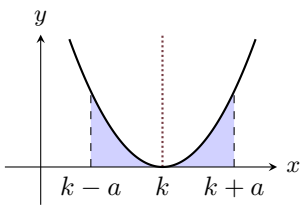
$$\int_{k-a}^{k+a} f(x)\,\mathrm{d}x = 0$$

Example: $(x - 1)^3$ in $x = 1$

$f(x)$ is **symmetric in** $x = k$ if

$$f(k + x) = f(k - x) \quad \text{for all real } x$$

that is, if $g(x) = f(x + k)$ is even: the line $x = k$ is a line of symmetry.



$$\int_{k-a}^{k+a} f(x)\,\mathrm{d}x = 2\int_{k}^{k+a} f(x)\,\mathrm{d}x$$

Example: $(x - 2)^2$ in $x = 2$

An interval from p to q is centred on $k = \frac{p+q}{2}$. Each boxed result needs its symmetry only on the interval of integration.