CORE PURE / MISCELLANEOUS PURE

Rational Functions

Mark scheme 14 questions · 177 marks

EXAM BOARDS AQA 7367 · CIE 9231

Questions

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	Total	**177**	

Reading the mark scheme

These rules apply throughout, unless a note says otherwise.

Marks

Every mark is worth one, so **M1 A1** on a row is two marks. Only marks coded ft follow through.

Code	Meaning
M1	Method: a correct method, applied to this question. A slip doesn't cost it if the method is clear. A formula must be used, not just quoted.
dM1	Dependent method: as M1, but only if the method mark before it was earned, unless its note names another.
A1	Accuracy: a correct answer or step. It needs the method mark it follows.
A1ft	As A1, following through from an earlier answer that the note names.
A1 AG	The last mark of a part whose answer is printed in the question: complete working, with no errors.
B1	A correct value, fact or statement. It needs no method mark.
B1ft	As B1, following through from an earlier answer that the note names.
E1	A written explanation, interpretation or conclusion. Concluding a proof by induction needs every earlier mark.

Abbreviations

ft and AG qualify a code in the Marks column. The others appear in the notes. A grey line gives the rule where the name alone doesn't.

ft	follow through Correct work from the student's own earlier wrong value earns the mark.	**AG**	answer given The answer is printed in the question, so every step to it must be shown.
cao	correct answer only No follow-through from an earlier error.	**cso**	correct solution only No error anywhere in the part, not just in its answer.
oe	or equivalent	**awrt**	answers which round to
soi	seen or implied Earned if written, or if later working shows it was used.	**isw**	ignore subsequent working A correct answer, once reached, isn't spoiled by later working.
SC	special case Credit, as the note says, for work outside the main method.	**their**	the student's own earlier value
condone	a slip that doesn't cost the mark	[ ]	working the mark doesn't need
s.f.	significant figures	**d.p.**	decimal places

Reading the scheme

- Each row is a line of working, with the marks it earns beside it. Where a row carries two marks, each note starts with its code.
- [$\boldsymbol{n}$] ends each part with its total.
- A shaded *Alternative* is a second method for the same total. Mark an answer by the method it follows.
- Any other valid method earns the equivalent marks, unless the question names a method or says *hence*.
- A sketch earns one B1 for each feature named below its figure.

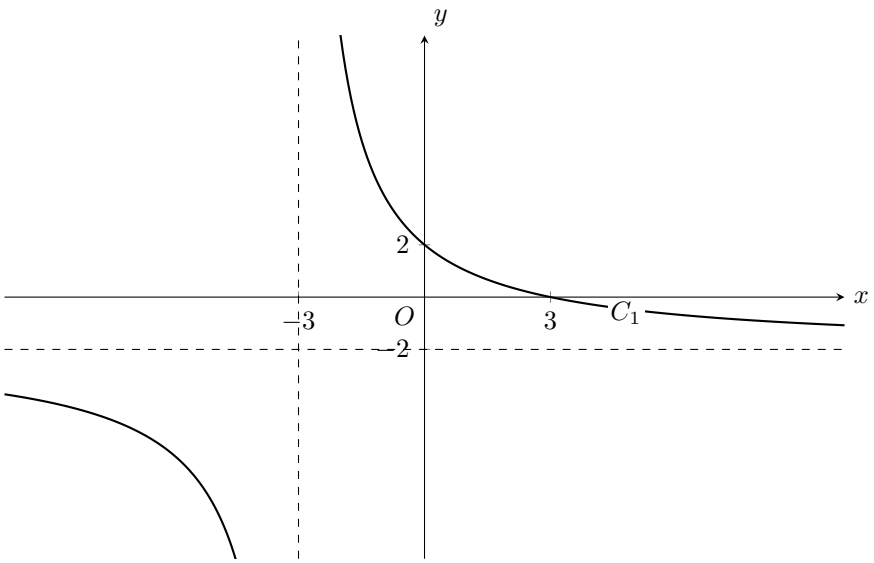
Accuracy and answers

- Unless a question says otherwise, give answers to 3 significant figures, and angles in degrees to 1 decimal place. Angles in radians take 3 significant figures.
- A final answer with more figures is accepted if it rounds to the scheme's value; one with fewer is not. An answer spoiled by rounding too early loses its accuracy mark.
- Where a question asks for an exact answer, a decimal loses the final accuracy mark. Otherwise either is accepted.
- Wrong or missing units cost nothing unless the question asks for them.
- An omitted $\mathrm{d}\theta$, $\mathrm{d}x$ or $\mathrm{d}u$ in an integral is not penalised where the variable of integration is clear from the integrand and the limits. It is penalised where it is genuinely ambiguous: a substitution part-way through, two variables in play, or limits in the wrong variable.
- A correct answer with no working earns full marks, unless the question needs the working: show that, prove, hence, a method it names, or an exact answer. A wrong answer with no working earns nothing. Calculator answers are treated the same way.
- A correct answer, once reached, isn't spoiled by later working (isw).

Value of g

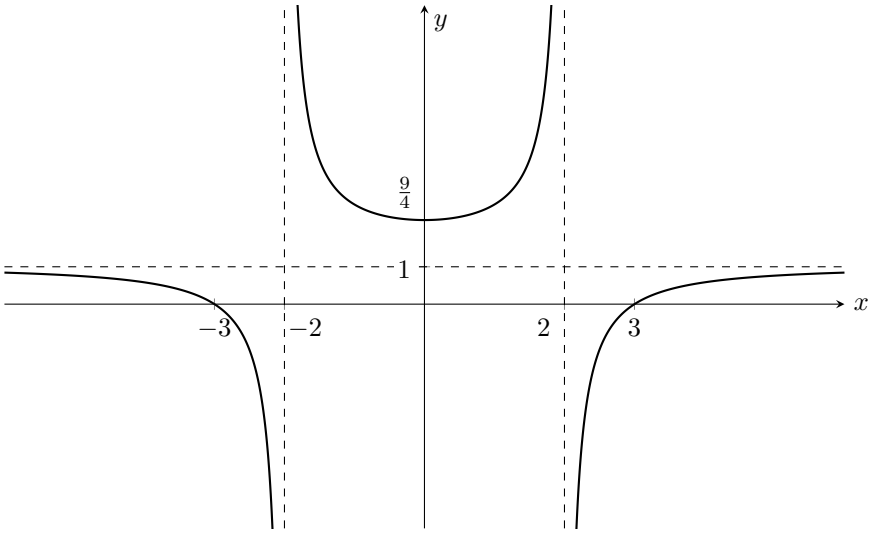
- Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question states a value, which is then the only one accepted. Answers that use $g = 9.8$ may be given to 2 or 3 significant figures.
- Where a question doesn't state g, an answer worked with $g = 10$ throughout earns the same marks, keeping any value printed in the question. The worked solutions list the final answers with $g = 10$.

Question	Scheme	Marks	Notes
1(a)	$y = 5$	**B1**	
		[1]	
1(b)	$x^2 + bx + a = (x+1)(x-2) = x^2 - x - 2$	**M1**	Compares coefficients of $x^2 + bx + a$ with $(x+1)(x-2)$, or substitutes $x = -1$ and $x = 2$ into $x^2 + bx + a = 0$ and solves. Condone sign slips.
	$a = -2,\ b = -1$	**A1**	Both values.
		[2]	
1(c)	$\left(0, -\frac{1}{2}\right)$	**B1**	Accept $y = -\frac{1}{2}$ stated with $x = 0$.
		[1]	
1(d)	$5x^2 - 2x + 1 = 4(x^2 - x - 2)$	**M1**	Equates their curve to 4 and multiplies by the denominator, collecting to a three-term quadratic. Condone sign slips.
	$x^2 + 2x + 9 = 0$	**A1**	oe, any nonzero multiple.
	$2^2 - 4(1)(9) = \ldots$	**M1**	Evaluates $b^2 - 4ac$ for their three-term quadratic, or completes the square. Condone a slip in the value.
	$= -32 < 0 \Rightarrow$ no real roots, so $y = 4$ does not intersect C	**A1 AG**	$-32 < 0$, or $(x+1)^2 + 8 > 0$, from the correct quadratic, with a conclusion that there are no real roots so no intersection.
		[4]	
Total for Question 1		**8**	

Question	Scheme	Marks	Notes
2(a)	$x = -3$	**B1**	
	$y = -2$	**B1**	
		[2]	
2(b)			
	Shape	**B1**	Two branches in the upper-right and lower-left regions, approaching the asymptotes without crossing them.
	Asymptotes	**B1**	Both asymptotes drawn and labelled $x = -3$, $y = -2$; values on the axes accepted.
	Intercepts	**B1**	Both $(0, 2)$ and $(3, 0)$, marked or stated; values on the axes accepted.
		[3]	
2(c)	Critical values $x = -3$, $x = 3$	**M1**	Uses the vertical asymptote and their x-intercept as critical values, from the sketch or algebraically, e.g. multiplying by $(x + 3)^2$.
	$-3 < x < 3$	**A1**	oe, e.g. set notation. A0 for $\leq$ at either end or any extra interval.
		[2]	
2(d)	$-x = -2 + \frac{12}{-y+3}$	**M1**	Replaces x by $-y$ and y by $-x$ in the equation of C_1. Not for swapping x and y without the sign changes.

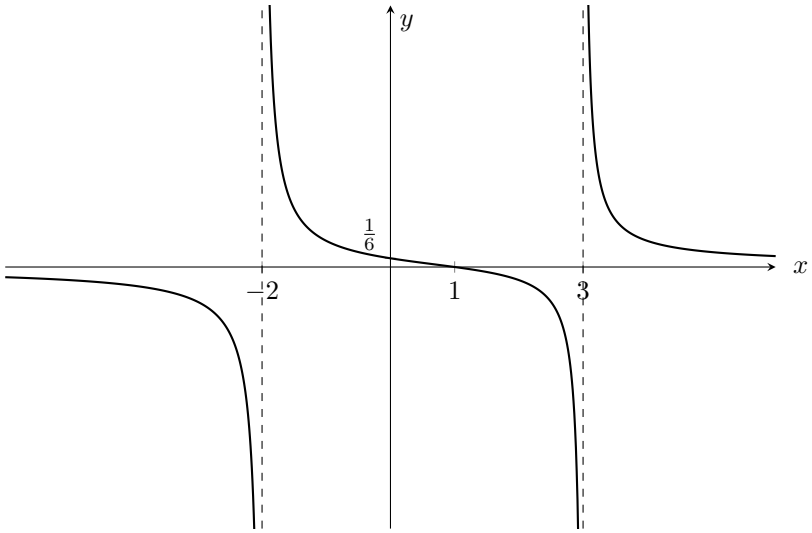
Question 2 continued

Question	Scheme	Marks	Notes
	$2-x=\frac{12}{3-y}$ $\Rightarrow 3-y=\frac{12}{2-x} \Rightarrow y=\ldots$	**dM1**	Rearranges their equation to make y the subject, clearing the fraction. Condone sign slips.
	$y=3+\frac{12}{x-2}$	**A1**	cso. oe in the form $y=f(x)$, e.g. $y=\frac{3x+6}{x-2}$.
		[3]	
	Total for Question 2	**10**	

Question	Scheme	Marks	Notes
3(a)	$x = 2,\ x = -2$	**B1**	Both needed.
	$y = 1$	**B1**	
		[2]	
3(b)			
	Shape	**B1**	U-shaped branch between the vertical asymptotes, tending to $+\infty$ at both, with its minimum on the positive y-axis.
	Shape	**B1**	Both outer branches rising from $-\infty$ at the vertical asymptotes, crossing the x-axis and approaching $y = 1$ from below.
	Asymptotes	**B1**	$x = \pm 2$ and $y = 1$ drawn, labelled by equations or values on the axes.
	Intercepts	**B1**	$(-3, 0)$ and $(3, 0)$ on the outer branches; values on the axis accepted.
	Intercepts	**B1**	$\left(0, \frac{9}{4}\right)$ or 2.25 marked on the central branch.
		[5]	
3(c)	Curve between $y = 0$ and $y = 1$: critical values $x = \pm 3 \Rightarrow x \ldots$	**M1**	Uses their x-intercepts as critical values and identifies where the curve lies between $y = 0$ and $y = 1$, from the sketch or algebraically.

Question 3 continued

Question	Scheme	Marks	Notes
	$x < -3$ or $x > 3$	**A1**	oe, such as $\|x\| > 3$ or set notation. Strict inequalities, with no other region.
		[2]	
	Total for Question 3	**9**	

Question	Scheme	Marks	Notes
4(a)(i)	$x = -2,\ x = 3$	**B1**	Both, and no other vertical asymptotes.
	$y = 0$	**B1**	
		[2]	
4(a)(ii)			
	Shape	**B1**	Left branch below the x-axis, to $-\infty$ at $x = -2$ and to 0 as $x \to -\infty$; right branch above, from $+\infty$ at $x = 3$ to 0.
	Shape	**B1**	Decreasing from $+\infty$ at $x = -2$ to $-\infty$ at $x = 3$, crossing the x-axis once.
	Asymptotes	**B1**	All three asymptotes shown: $x = -2$, $x = 3$, $y = 0$, labelled by equations or values on the axes.
	Intercepts	**B1**	Both $(1, 0)$ and $\left(0, \frac{1}{6}\right)$, marked or stated; values on the axes accepted.
		[4]	
4(b)	$\frac{x-1}{x^2-x-6} = \frac{1}{2} \Rightarrow x^2 - 3x - 4 = 0$ $\Rightarrow x = \ldots$	**M1**	Equates the curve to $\frac{1}{2}$, clears fractions to a three-term quadratic and solves it by any means. Condone sign slips in rearranging.
	$x = -1,\ x = 4$	**A1**	

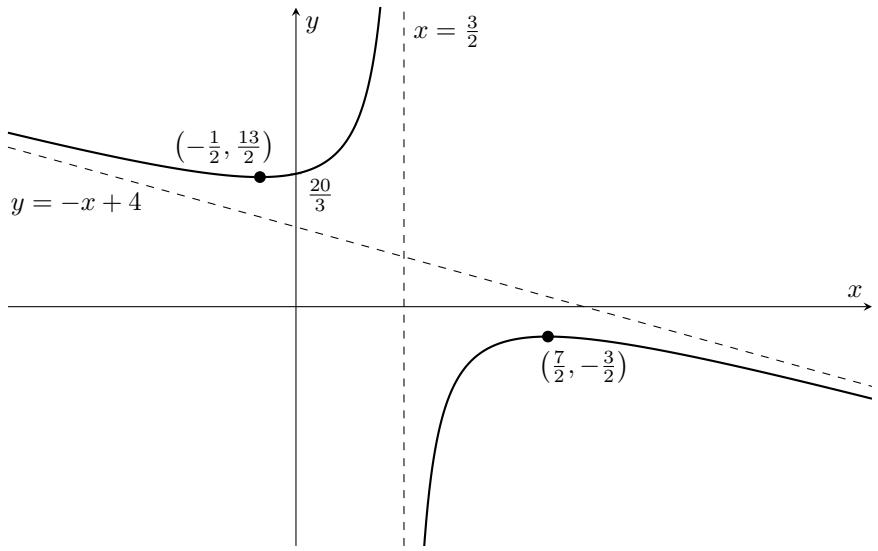
Question 4 continued

Question	Scheme	Marks	Notes
	$-2 < x < -1$ or $3 < x < 4$	**A1**	Both intervals with strict inequalities, oe in set notation; condone "and" between them. A0 if any extra interval is given.
		[3]	
	Total for Question 4	**9**	

Question	Scheme	Marks	Notes
5(a)	$y = 2$	**B1**	Must be an equation, not 2 alone.
	$x = -1$	**B1**	Must be an equation, not -1 alone.
		[2]	
5(b)	$3x + c = 2 - \frac{3}{x+1}$ $\Rightarrow (3x + c - 2)(x + 1) = -3$	**M1**	Equates line and curve and multiplies through by $x + 1$. Condone sign slips.
	$3x^2 + 3x + (c - 2)x + c - 2 = -3$	**A1**	Correct expanded equation with no fractions, any arrangement; terms need not be collected.
	$3x^2 + (c + 1)x + (c + 1) = 0$	**A1 AG**	From a correct expanded equation, reaches the printed result, including $= 0$, with no errors; collected steps accepted.
		[3]	
5(c)(i)	$b^2 - 4ac = 0$: $(c + 1)^2 - 12(c + 1) = 0$	**M1 A1**	**M1** Sets the discriminant of the quadratic in (b) to zero, with its coefficients substituted. Condone one slip. Not for a calculus method. **A1** oe, e.g. $c^2 - 10c - 11 = 0$.
	$c = -1$ or $c = 11$	**A1**	Both values and no others.
		[3]	
5(c)(ii)	$x = -\frac{c+1}{2\times 3}$ with $c =$ " -1", "11" $\Rightarrow x = \ldots$	**M1**	Finds the repeated root for at least one of their values of c, by $x = -\frac{b}{2a}$ or by solving the quadratic in (b).
	$x = 0$, $x = -2$	**A1**	Both values.
	$y = 3(0) - 1$, $y = 3(-2) + 11$	**dM1**	Substitutes their x with its corresponding c into the line, or substitutes x into the curve, to find y.
	$(0, -1)$ and $(-2, 5)$	**A1**	Both points, correctly paired.
		[4]	
Total for Question 5		**12**	

Question	Scheme	Marks	Notes
6(a)	$y(2x-3) = -2x^2 + 11x - 20$	**M1**	Multiplies by $2x-3$ and rearranges towards a quadratic in x with coefficients in y. Condone sign slips.
	$2x^2 + (2y-11)x + (20-3y) = 0$	**A1**	oe, terms collected.
	$(2y-11)^2 - 4(2)(20-3y) \geq 0$	**M1**	Applies $b^2 - 4ac$ to their quadratic in x, with $\geq$, $>$ or $=$, giving a quadratic in y. Condone sign slips.
	$4y^2 - 20y - 39 \geq 0 \Rightarrow y = \ldots$ $[= -\frac{3}{2},\ \frac{13}{2}]$	**dM1**	Solves their three-term quadratic in y for its critical values by any means, calculator allowed.
	$f(x) \leq -\frac{3}{2}$ or $f(x) \geq \frac{13}{2}$	**A1**	Both non-strict inequalities, oe in set or interval notation; y for $f(x)$ condoned. No check of endpoints or $x = \frac{3}{2}$ needed. Accept the range from a correct sketch in (d).
		[5]	
6(b)	$f'(x) = -1 + \frac{16}{(2x-3)^2}$	**M1**	Differentiates by the quotient rule or after dividing. Condone sign and coefficient slips.
	$f'(x) = 0 \Rightarrow (2x-3)^2 = 16 \Rightarrow x = \ldots$ $\Rightarrow y = \ldots$	**dM1**	Sets their derivative to zero, solves for x and substitutes at least one root to find y.
	$\left(-\frac{1}{2}, \frac{13}{2}\right)$, $\left(\frac{7}{2}, -\frac{3}{2}\right)$	**A1**	Both points, correctly paired.
	Alternative: Repeated roots from (a)		
	$2x^2 + (2y-11)x + (20-3y) = 0$ $y = \frac{13}{2} \Rightarrow (2x+1)^2 = 0$ $y = -\frac{3}{2} \Rightarrow (2x-7)^2 = 0$	**M1**	Substitutes the critical values from (a) into their quadratic in x. Condone sign slips.
	$x = -\frac{1}{2}$ or $x = \frac{7}{2}$ Attained range endpoints are extrema, hence stationary points.	**dM1**	Solves for the repeated roots and identifies the attained range endpoints as differentiable extrema, hence stationary points.
	$\left(-\frac{1}{2}, \frac{13}{2}\right)$, $\left(\frac{7}{2}, -\frac{3}{2}\right)$	**A1**	Both points, correctly paired.
		[3]	
6(c)	$f(x) = -x + 4 - \frac{8}{2x-3}$	**M1**	Divides to the form $ax + b + \frac{c}{2x-3}$ by any method; may be seen in (b). Condone arithmetic slips.

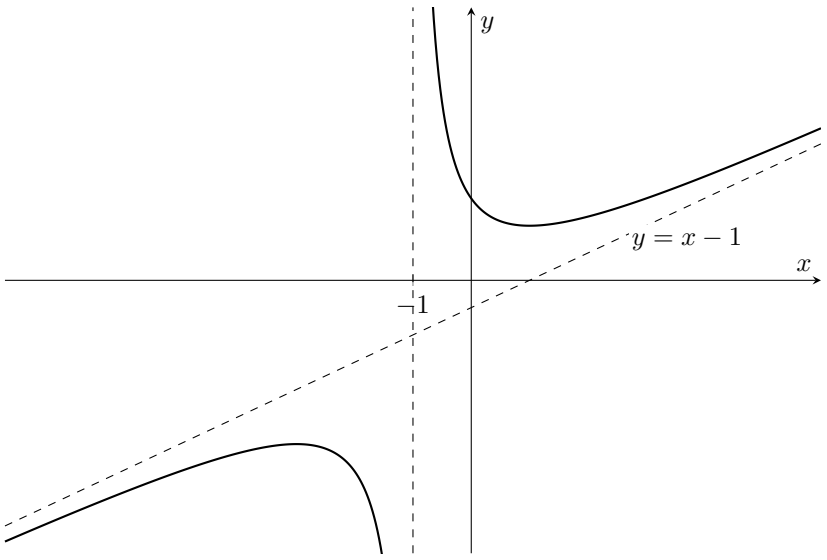
Question 6 continued

Question	Scheme	Marks	Notes
	$y = -x + 4$	**A1**	From a correct quotient; a slip in the remainder alone is condoned.
		[2]	
6(d)			
	Shape	**B1**	Left branch left of the vertical asymptote and above the oblique; right branch right of the vertical and below the oblique. Both approach the asymptotes, with no x-intercepts.
	Asymptotes	**B1**	Vertical asymptote at $x = \frac{3}{2}$, labelled.
	Asymptotes	**B1**	Oblique asymptote with negative gradient, labelled $y = -x + 4$.
	Turning points	**B1**	Both points labelled with coordinates: minimum $\left(-\frac{1}{2}, \frac{13}{2}\right)$ on the left branch, maximum $\left(\frac{7}{2}, -\frac{3}{2}\right)$ on the right.
		[4]	
	Total for Question 6	**14**	

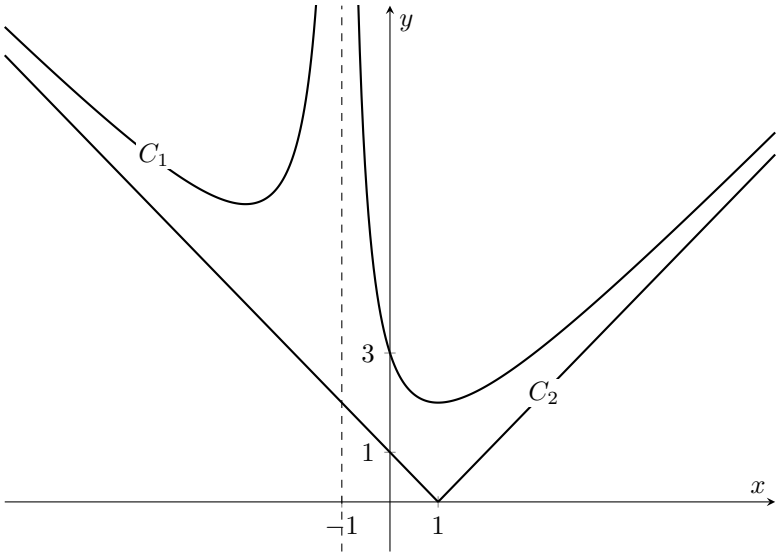
Question	Scheme	Marks	Notes
7(a)	$y = 1$	**B1**	
		[1]	
7(b)	$p^2 - 4(1)(4) < 0 \Rightarrow p^2 - 16 < 0$	**M1 A1**	**M1** Uses the discriminant of the denominator, set negative. Condone $\leq$ for this mark. **A1** $p^2 - 16 < 0$ oe, such as $p^2 < 16$.
	$p^2 = 16 \Rightarrow p = \ldots$, inside region	**dM1**	Finds the critical values of their quadratic inequality and selects the inside region.
	$-4 < p < 4$	**A1**	Strict inequalities; accept $\{p : -4 < p < 4\}$ or $(-4, 4)$.
		[4]	
7(c)	$x^2 + 2x - 3 = 0 \Rightarrow x = \ldots$	**M1**	Sets the numerator equal to zero and solves by any means.
	$(-3, 0)$, $(1, 0)$	**A1**	Both points, and no others.
	$\left(0, -\frac{3}{4}\right)$	**B1**	
		[3]	
7(d)(i)	$x^2 + 2x - 3 = k(x^2 - x + 4)$	**M1**	Sets the curve equal to k, clears the fraction and collects into a three-term quadratic in x. Condone sign slips.
	$(1 - k)x^2 + (2 + k)x - (3 + 4k) = 0$	**A1**	Correct quadratic, oe with all terms on one side.
	$(2 + k)^2 - 4(1 - k)\big(-(3 + 4k)\big) \geq 0$	**M1**	Uses $b^2 - 4ac$ with their coefficients in k, set non-negative. Condone $>$ or $=$ for this mark.
	$\Rightarrow -15k^2 + 8k + 16 \geq 0$ $\Rightarrow 15k^2 - 8k - 16 \leq 0$	**A1 AG**	Correct substituted discriminant with ≥ 0, expanded without error to the printed result, the inequality reversed correctly on changing signs.
		[4]	
7(d)(ii)	$15k^2 - 8k - 16 = 0 \Rightarrow k = \ldots$	**M1**	Solves $15k^2 - 8k - 16 = 0$ by any means, calculator allowed.

Question 7 continued

Question	Scheme	Marks	Notes
	$k = \frac{4}{3}$ or $-\frac{4}{5}$ y-coordinate of M is $\frac{4}{3}$	**A1**	$\frac{4}{3}$ or awrt 1.33 given as the answer; A0 if $-\frac{4}{5}$ is also given as the y-coordinate.
		[2]	
	Total for Question 7	**14**	

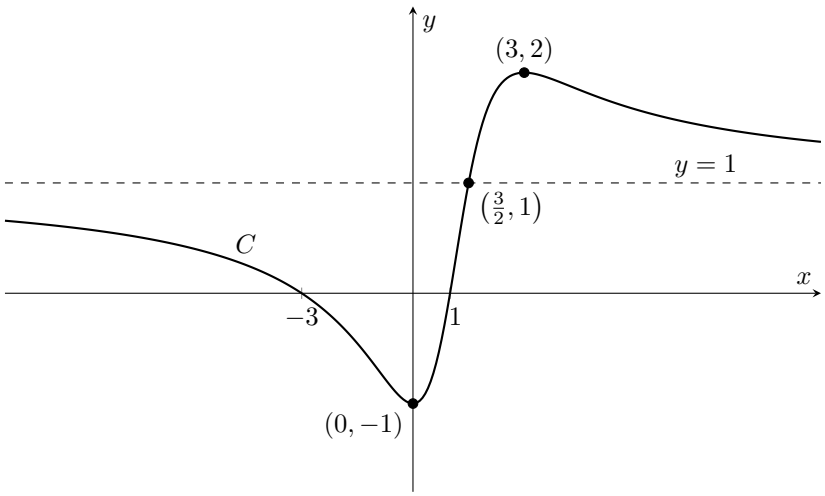
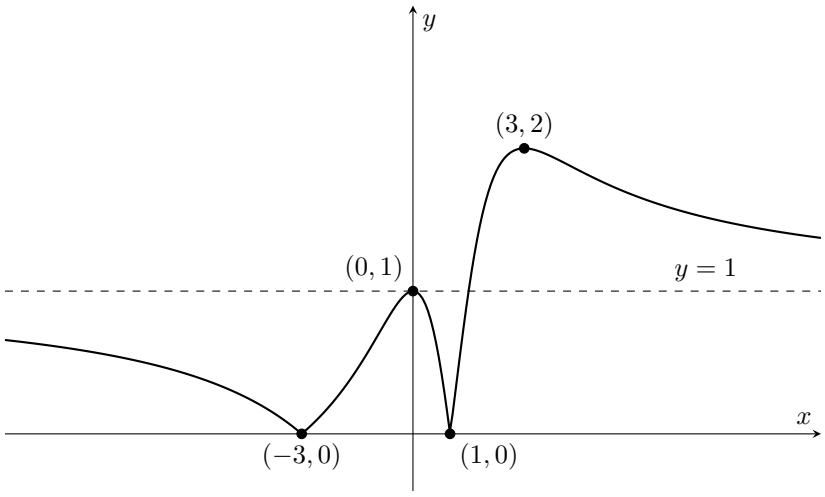
Question	Scheme	Marks	Notes
8(a)	$x = -1$	**B1**	
	$\frac{x^2+3}{x+1} = x - 1 + \frac{4}{x+1}$	**M1**	Divides to the form $x + p + \frac{q}{x+1}$ by any method. Condone slips in p and q.
	$y = x - 1$	**A1**	
		[3]	
8(b)	$y(x+1) = x^2 + 3$	**M1**	Multiplies by $x + 1$ and rearranges towards a quadratic in x with coefficients in y. Condone sign slips.
	$x^2 - yx + (3 - y) = 0$	**A1**	oe, any arrangement equated to zero.
	Real x needs $b^2 - 4ac \geq 0$: $(-y)^2 - 4(3 - y) \geq 0$ $\Rightarrow y^2 + 4y - 12 \geq 0$	**M1**	Uses the discriminant of their quadratic in x to form a quadratic inequality or equation in y. Condone slips and a wrong inequality sign.
	$(y + 6)(y - 2) \geq 0 \Rightarrow y = \ldots$ $[= -6,\ 2]$	**dM1**	Solves their three-term quadratic in y by factorising, formula, completing the square or calculator.
	$y \leq -6$ or $y \geq 2$ so no point on C with $-6 < y < 2$	**A1 AG**	cso: correct region from $b^2 - 4ac \geq 0$, or discriminant < 0 shown for $-6 < y < 2$, with a conclusion in words. A0 for the region alone.
		[5]	
8(c)			

Question 8 continued

Question	Scheme	Marks	Notes
	Shape	**B1**	Two branches approaching both asymptotes without crossing them; right branch above the x-axis, left branch below it.
	Asymptotes	**B1**	Both asymptotes drawn and labelled $x = -1$, $y = x - 1$. Intercept and turning-point labels are optional.
		[2]	
8(d)(i)			
	Shape	**B1**	C_1: right branch unchanged, left branch reflected above the x-axis; the whole curve is above the axis.
	Asymptotes	**B1**	C_1 approaches $x = -1$, $y = 1 - x$ on the left and $y = x - 1$ on the right; C_2 may supply the oblique asymptotes.
	Shape	**B1**	C_2: V with vertex at $(1, 0)$, arms of gradients -1 and 1; C_1 stays above it.
	Intercepts	**B1**	$(0, 3)$ on C_1, and both $(1, 0)$, $(0, 1)$ on C_2, marked or stated; axis values accepted.
		[4]	
8(d)(ii)	$c \leq 0$	**B1**	oe; the inequality must include 0.
		[1]	
	Total for Question 8	**15**	

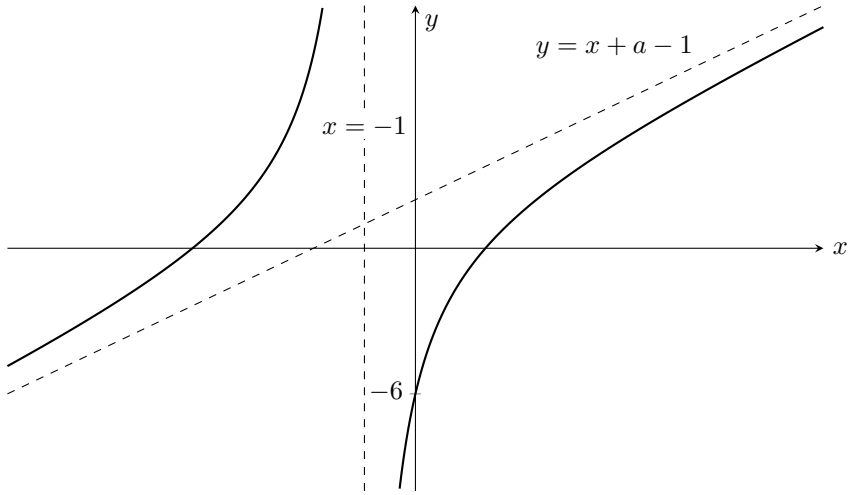
Question	Scheme	Marks	Notes
9(a)	$y = 1$	**B1**	
	$(-2)^2 - 4(1)(3) = -8$	**M1**	Finds the discriminant of the denominator to examine whether it has real roots. Condone arithmetic slips.
	$-8 < 0 \Rightarrow$ denominator never zero $\Rightarrow$ no vertical asymptotes	**A1 AG**	Correct discriminant $-8 < 0$ and conclusion that the denominator is never zero, hence no vertical asymptotes.
	Alternative: Completing the square		
	$y = 1$	**B1**	
	$x^2 - 2x + 3 = (x-1)^2 + 2$	**M1**	Completes the square on the denominator to examine whether it has real roots.
	$(x-1)^2 + 2 > 0$ for all real x $\Rightarrow$ denominator never zero $\Rightarrow$ no vertical asymptotes	**A1 AG**	Correct positive square form for all real x, with the conclusion that the denominator is never zero, hence no vertical asymptotes.
		[3]	
9(b)	$\frac{dy}{dx} = \{(2x+2)(x^2-2x+3)$ $-(x^2+2x-3)(2x-2)\} \div (x^2-2x+3)^2$	**M1 A1**	**M1** Applies the quotient or product rule, with linear derivatives of numerator and denominator. Condone sign slips. **A1** Any correct form, simplified or not.
	$12x - 4x^2 = 0 \Rightarrow x = \ldots$ $\Rightarrow y = \ldots$	**dM1**	Sets their numerator equal to zero, solves by any method and substitutes at least one root into y.
	$(0, -1)$ and $(3, 2)$	**A1**	Both points and no others.
		[4]	

Question 9 continued

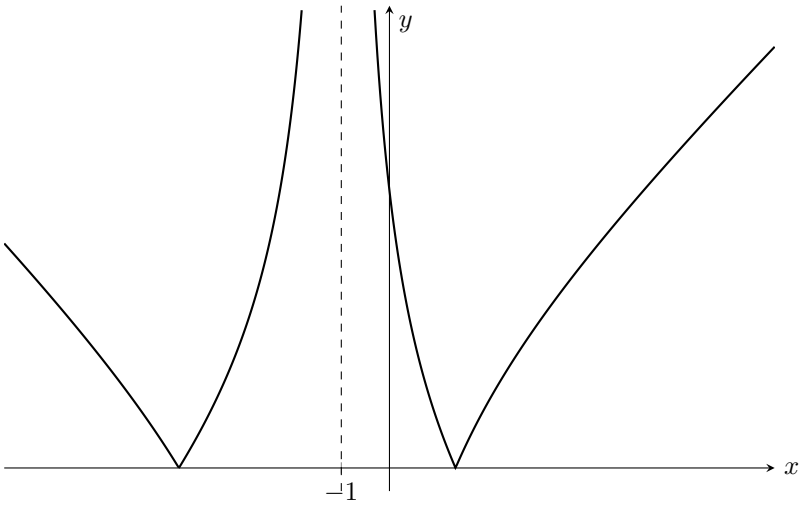
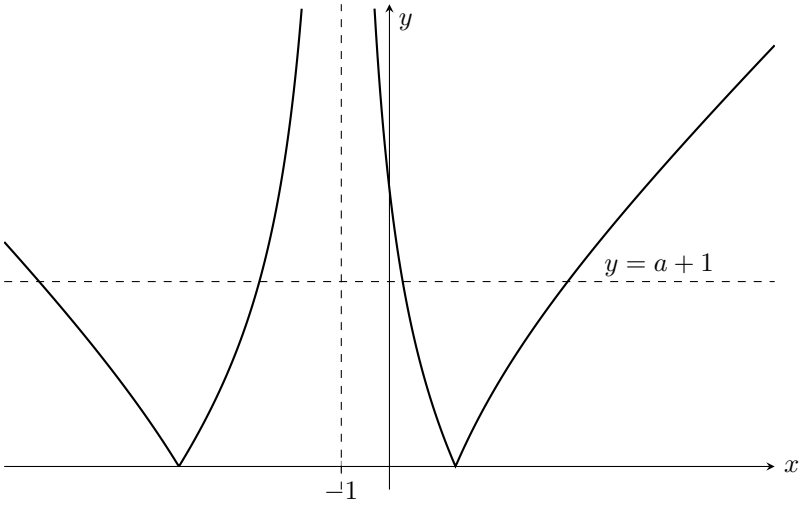
Question	Scheme	Marks	Notes
9(c)			
	Shape	**B1**	One continuous curve: minimum on the negative y-axis, maximum in the first quadrant; both ends approach the horizontal asymptote, from below on the left and above on the right.
	Asymptotes	**B1**	Horizontal asymptote $y = 1$ drawn and labelled; 1 on the y-axis accepted.
	Intercepts	**B1**	All three $(-3, 0)$, $(1, 0)$, $(0, -1)$, marked or stated; values on the axes accepted.
		[3]	
9(d)			
	Shape	**B1**	Part below the x-axis reflected above it, with cusps at $x = -3$ and $x = 1$, and a local maximum on the positive y-axis.

Question 9 continued

Question	Scheme	Marks	Notes
	$0 < k < 1$	**B1**	Strict inequalities, oe such as $k \in (0, 1)$.
		[2]	
	Total for Question 9	**12**	

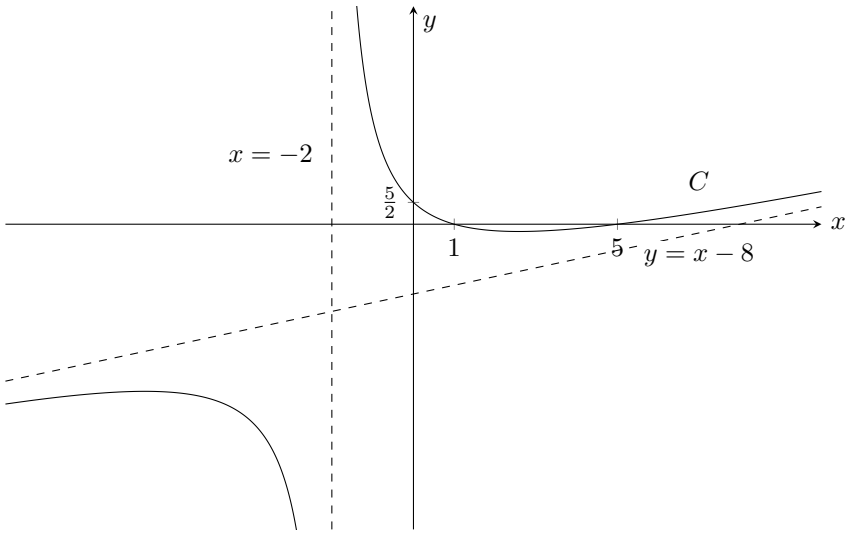
Question	Scheme	Marks	Notes
10(a)	$x = -1$	**B1**	
	$\frac{x^2+ax-6}{x+1} = x + \ldots + \frac{\ldots}{x+1}$	**M1**	Divides by $x+1$ by any method, e.g. comparing coefficients, reaching a linear quotient $x + \ldots$. Condone a wrong remainder.
	$y = x + a - 1$	**A1**	Must be an equation; $x + a - 1$ alone scores A0.
		[3]	
10(b)	$y = x + a - 1 - (a+5)(x+1)^{-1}$ $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \ldots$	**M1**	Differentiates their (a) form with $(x+1)^{-1} \to k(x+1)^{-2}$, or the original by the quotient rule with denominator $(x+1)^2$. Condone sign slips.
	$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{a+5}{(x+1)^2}$	**A1**	oe, e.g. $\frac{x^2+2x+a+6}{(x+1)^2}$.
	$a > 0 \Rightarrow \frac{a+5}{(x+1)^2} > 0$	**dM1**	Considers the sign of their derivative, or sets it to zero and tests for real roots, e.g. by discriminant or completing the square.
	$\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ for all $x \neq -1$, so no stationary points	**A1 AG**	Correct derivative shown nonzero for all x, using $a > 0$ or $a + 5 > 0$, with a conclusion in words. A0 without reference to a.
		[4]	
10(c)			
	Shape	**B1**	Two increasing branches, left above and right below the oblique asymptote, approaching both asymptotes, each crossing the x-axis once.

Question 10 continued

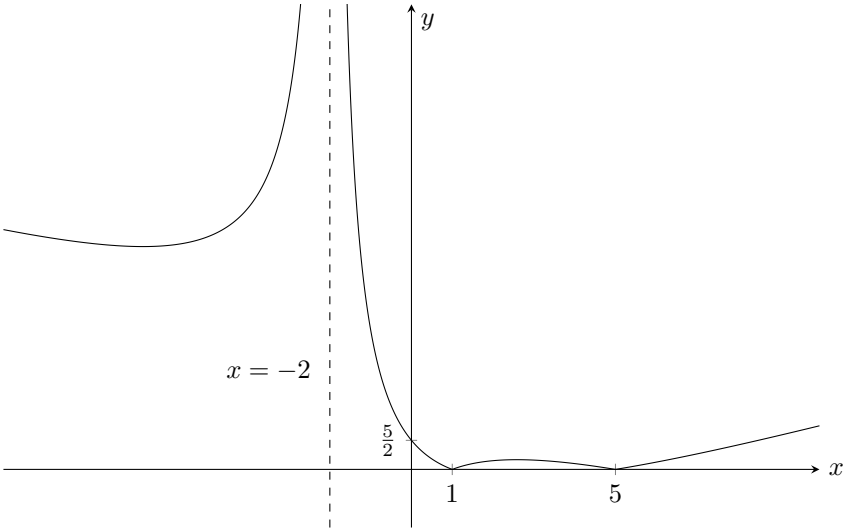
Question	Scheme	Marks	Notes
	Asymptotes	**B1**	Both asymptotes drawn and labelled $x = -1$ and $y = x + a - 1$.
	Intercepts	**B1**	$(0, -6)$ stated, or -6 marked on the y-axis.
		[3]	
10(d)(i)			
	Shape	**B1**	Parts of C below the x-axis reflected above it; both branches tend to $+\infty$ at $x = -1$, with both ends rising.
	Intercepts	**B1**	Cusps on the x-axis, one on each side of $x = -1$.
		[2]	
10(d)(ii)			

Question 10 continued

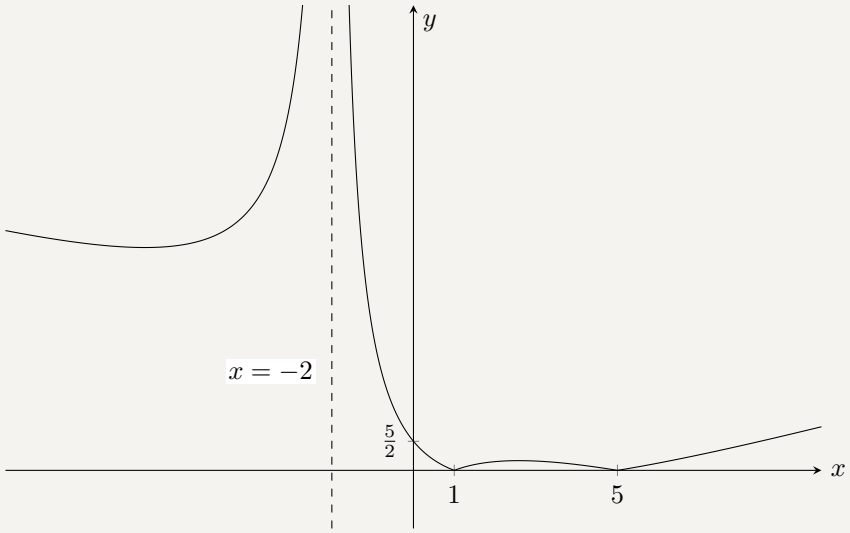
Question	Scheme	Marks	Notes
	Intercepts	**B1**	Horizontal line meeting the curve four times, twice on each side of $x = -1$.
		[1]	
10(d)(iii)	$x = 0$ is a boundary: $\left\lvert\frac{-6}{1}\right\rvert = a + 1 \Rightarrow a = \ldots$	**M1**	Substitutes an endpoint of an interval, e.g. $x = 0$, into $\left\lvert\frac{x^2+ax-6}{x+1}\right\rvert = a + 1$ and solves for a.
	$a = 5$	**A1**	$a = 5$ only, with any other root from the modulus rejected.
		[2]	
	Total for Question 10	**15**	

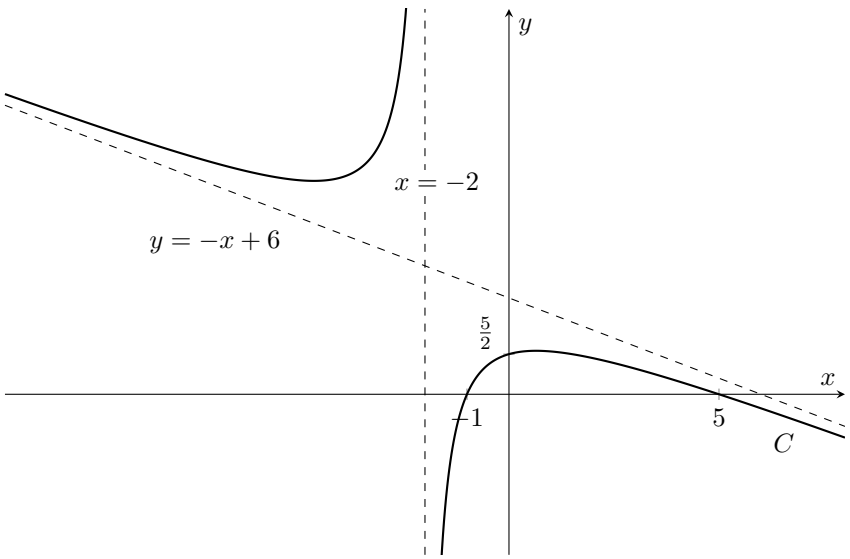
Question	Scheme	Marks	Notes
11(a)	$x = -2$	**B1**	
	$x^2 - 6x + 5 = (x+2)(x+\ldots) + \ldots$	**M1**	Divides by $x+2$ by any method to a quotient of the form $x+k$. Condone an arithmetic slip in k or the remainder.
	$y = x - 8$	**A1**	
		[3]	
11(b)	$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \frac{21}{(x+2)^2}$	**M1 A1**	**M1** Differentiates their divided form with $(x+2)^{-1} \to k(x+2)^{-2}$, or uses the quotient rule with numerator of the form $(x+2)(2x-6) \pm (x^2-6x+5)$. **A1** oe, such as $\frac{x^2+4x-17}{(x+2)^2}$.
	$1 - \frac{21}{(x+2)^2} = 0 \Rightarrow x = \ldots$ $y = x - 8 + \frac{21}{x+2} \Rightarrow y = \ldots$	**dM1**	Sets their derivative to zero, solves for x by any method and substitutes at least one value into an equation of C.
	$(-2-\sqrt{21},\ -10-2\sqrt{21})$ $(-2+\sqrt{21},\ -10+2\sqrt{21})$	**A1**	Both points, exact and correctly paired, in any exact equivalent form.
		[4]	
11(c)			
	Shape	**B1**	Two branches approaching both asymptotes: left branch with a maximum below the x-axis, right branch with a minimum below it.
	Asymptotes	**B1**	Both asymptotes drawn and labelled $x = -2$, $y = x - 8$.

Question 11 continued

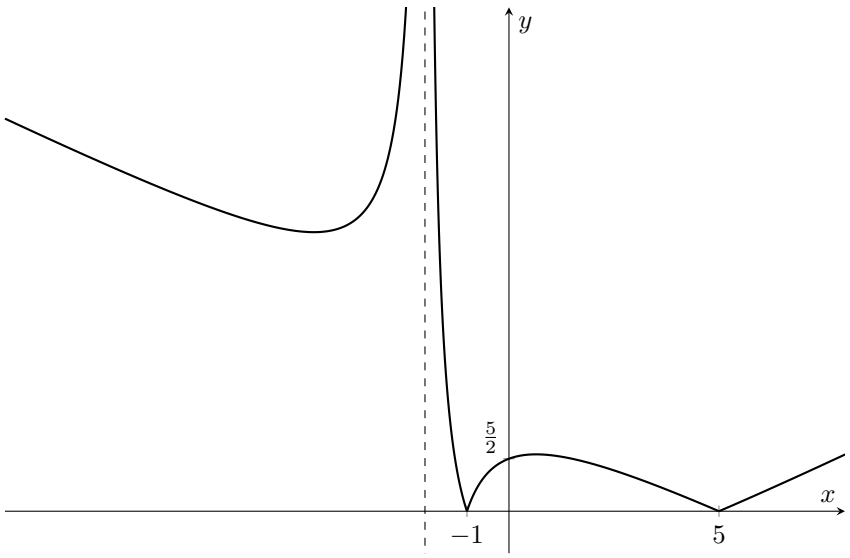
Question	Scheme	Marks	Notes
	Intercepts	**B1**	All three $(1,0)$, $(5,0)$, $\left(0,\frac{5}{2}\right)$, marked or stated; values on the axes accepted.
		[3]	
11(d)			
	Shape	**B1**	Left branch and the arc between $x=1$ and $x=5$ reflected above the x-axis, cusps at $x=1$ and $x=5$, asymptote $x=-2$; nothing below the x-axis.
	$\frac{x^2-6x+5}{x+2}=6$ $\Rightarrow x^2-6x+5=6(x+2)$	**M1**	Equates the original curve to 6 and clears the fraction. Condone sign slips.
	$x^2-12x-7=0$ $y=-6$ does not meet C, from the sketch or (b).	**A1**	Correct quadratic, with intersections at $y=-6$ dismissed from the sketch or the extrema in (b).
	$x^2-12x-7=0 \Rightarrow x=\ldots$	**dM1**	Solves their three-term quadratic by any method to $x=\ldots$.
	$6-\sqrt{43}<x<6+\sqrt{43}$	**A1**	Correct interval read from the modulus sketch, exact in any equivalent notation. Strict inequalities; no extra interval.

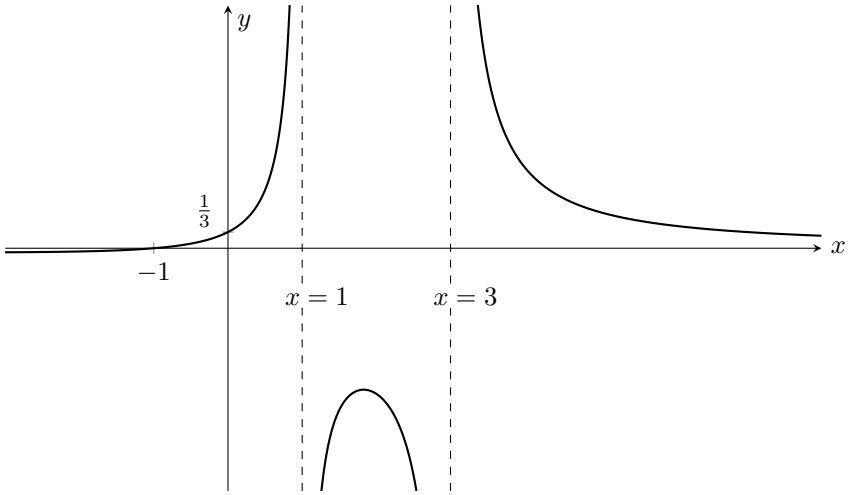
Question 11 continued

Question	Scheme	Marks	Notes
	Alternative: Squaring		
			
	Shape	**B1**	Left branch and the arc between $x = 1$ and $x = 5$ reflected above the x-axis, cusps at $x = 1$ and $x = 5$, asymptote $x = -2$; nothing below the x-axis.
	$(x^2 - 6x + 5)^2 < 36(x + 2)^2$ $\Rightarrow \big(x^2 - 6x + 5 - 6(x + 2)\big)$ $\quad \times \big(x^2 - 6x + 5 + 6(x + 2)\big) < 0$	**M1**	Squares, clears $(x + 2)^2$ and factorises as a difference of two squares, or forms both $x^2 - 6x + 5 = \pm 6(x + 2)$. Condone sign slips.
	$(x^2 - 12x - 7)(x^2 + 17) < 0$ $\Rightarrow x^2 - 12x - 7 < 0$	**A1**	$x^2 - 12x - 7 < 0$, or $x^2 - 12x - 7 = 0$ as the only critical equation, with $x^2 + 17$ rejected.
	$x^2 - 12x - 7 = 0 \Rightarrow x = \ldots$	**dM1**	Solves their three-term quadratic by any method to $x = \ldots$.
	$6 - \sqrt{43} < x < 6 + \sqrt{43}$	**A1**	Any exact equivalent, or set notation. Strict inequalities; A0 with any extra interval.
		[5]	
	Total for Question 11	**15**	

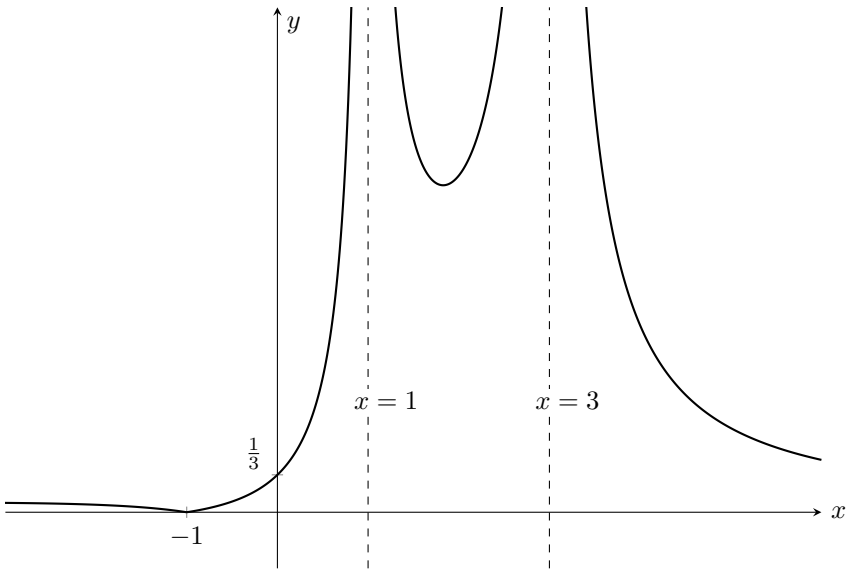
Question	Scheme	Marks	Notes
12(a)	$x = -2$	**B1**	
	$y = -x + 6 - \dfrac{7}{x+2}$	**M1**	Divides to the form $px + q + \frac{r}{x+2}$, by long division or comparing coefficients. Condone slips in values.
	$y = -x + 6$	**A1**	
		[3]	
12(b)	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -1 + \dfrac{7}{(x+2)^2}$	**M1 A1**	**M1** Differentiates their divided form, power of at least one term decreased by 1, or applies the quotient rule to the original. Condone sign slips. **A1** oe, such as $\frac{-x^2-4x+3}{(x+2)^2}$.
	$-1 + \dfrac{7}{(x+2)^2} = 0 \Rightarrow x = \ldots$ $\Rightarrow y = \ldots$	**dM1**	Sets their derivative to zero, solves for x by any means and substitutes at least one value to find y.
	$(-2-\sqrt{7},\ 8+2\sqrt{7}),\ (-2+\sqrt{7},\ 8-2\sqrt{7})$	**A1**	Both points, exact or awrt $(-4.65, 13.3)$ and $(0.646, 2.71)$.
		[4]	
12(c)			
	Shape	**B1**	Left branch with a minimum, right branch with a maximum; both approach the asymptotes, with the left tending to $+\infty$ and right to $-\infty$ at the vertical asymptote.
	Asymptotes	**B1**	Both asymptotes drawn and labelled $x = -2$, $y = -x + 6$.

Question 12 continued

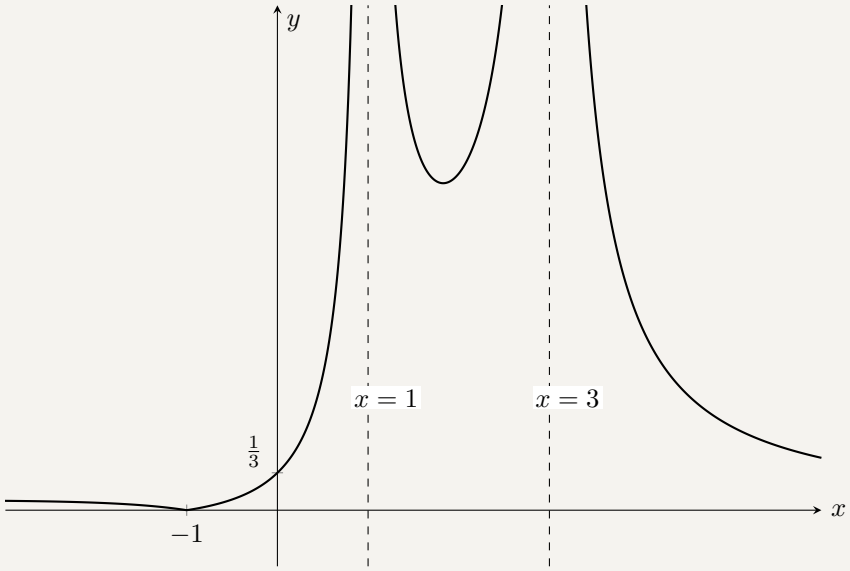
Question	Scheme	Marks	Notes
	Intercepts	**B1**	All three $(-1,0)$, $(5,0)$, $\left(0,\frac{5}{2}\right)$, marked or stated; values on the axes accepted.
		[3]	
12(d)			
	Shape	**B1**	Parts of C below the x-axis reflected above it, the rest unchanged; both sides tend to $+\infty$ at $x=-2$.
	Intercepts	**B1**	Cusps on the x-axis at $x=-1$ and $x=5$.
		[2]	
12(e)	$-x^2+4x+5=2(x+2)\Rightarrow x=\ldots$ $-x^2+4x+5=-2(x+2)\Rightarrow x=\ldots$	**M1**	Forms quadratics from both $f(x)=2$ and $f(x)=-2$, or a squared quartic, and solves them by any means. Condone slips.
	$x=1\pm\sqrt{2}$	**A1**	Exact, or awrt 2.41 and -0.414.
	$x=3\pm3\sqrt{2}$	**A1**	Exact, or awrt 7.24 and -1.24.
	$3-3\sqrt{2}<x<1-\sqrt{2}$ or $1+\sqrt{2}<x<3+3\sqrt{2}$	**A1**	Both intervals, strict inequalities, any correct notation; exact or awrt. A0 for any extra interval.
		[4]	
	Total for Question 12	**16**	

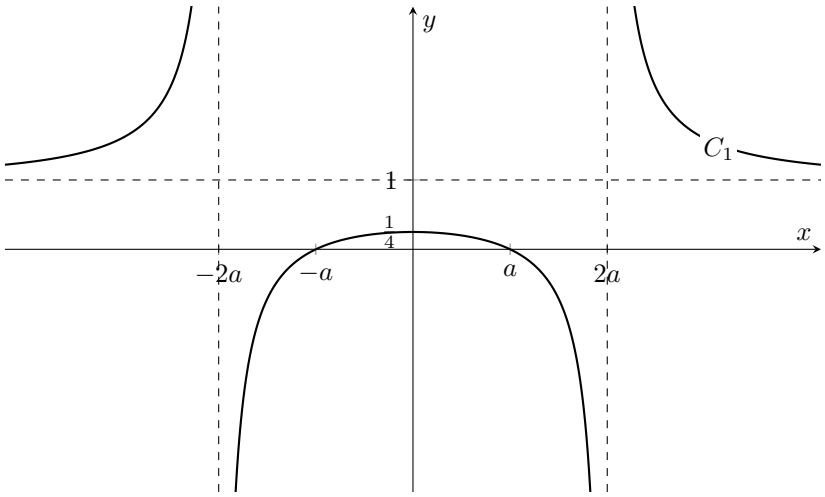
Question	Scheme	Marks	Notes
13(a)	$x = 1,\ x = 3$	**B1**	Both required.
	$y = 0$	**B1**	
		[2]	
13(b)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x^2-4x+3)-(x+1)(2x-4)}{(x^2-4x+3)^2}$	**M1**	Applies the quotient rule, product rule or equivalent, with terms in $x^2 - 4x + 3$ and $(x + 1)(2x - 4)$ in the numerator. Condone a sign slip.
	$= \frac{7-2x-x^2}{(x^2-4x+3)^2}$	**A1**	Any equivalent form; need not be simplified.
	$7 - 2x - x^2 = 0 \Rightarrow x = \ldots\ [= -1 \pm 2\sqrt{2}]$ $\Rightarrow y = \ldots$	**dM1**	Sets their numerator to zero, solves the quadratic by any means, and substitutes at least one root into y.
	$\left(-1 - 2\sqrt{2},\ \sqrt{2} - \frac{3}{2}\right)$ $\left(-1 + 2\sqrt{2},\ -\sqrt{2} - \frac{3}{2}\right)$	**A1**	Both points correctly paired, exact in any form or awrt $(-3.83, -0.0858)$ and $(1.83, -2.91)$.
		[4]	
13(c)			
	Shape	**B1**	Three branches: left with a negative minimum then rising to $+\infty$; middle below the axis with a maximum; right positive tending to zero.
	Asymptotes	**B1**	All three asymptotes drawn and labelled $x = 1$, $x = 3$, $y = 0$; the x-axis may represent $y = 0$.

Question 13 continued

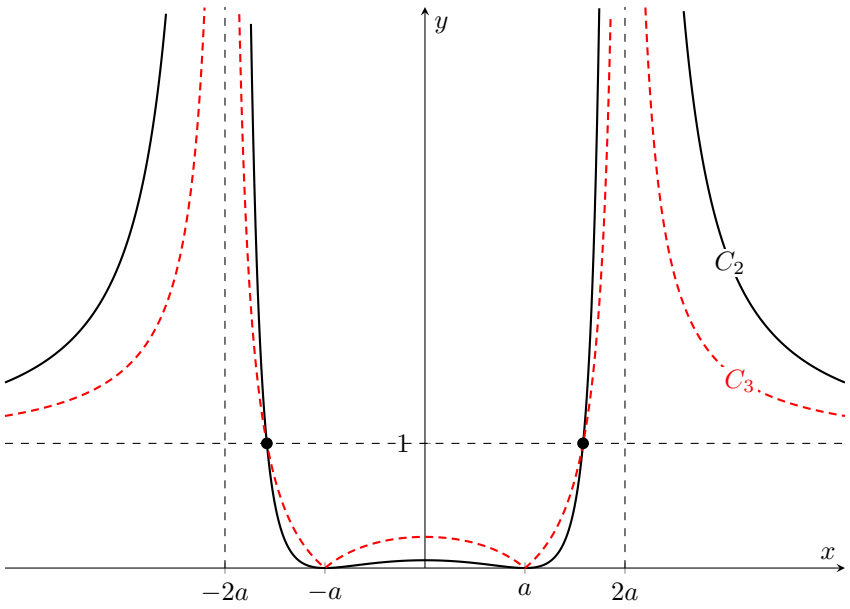
Question	Scheme	Marks	Notes
	Intercepts	**B1**	Both $(-1, 0)$ and $\left(0, \frac{1}{3}\right)$, marked or stated; values on the axes accepted.
		[3]	
13(d)			
	Shape	**B1**	Curve from (c) with negative parts reflected: all on or above the x-axis, with a cusp on the x-axis at $x = -1$ and asymptotes $x = 1$, $x = 3$, $y = 0$.
	$x \neq 1, 3$: $\left\lvert \frac{x+1}{x^2-4x+3} \right\rvert > \frac{1}{2}$ $\frac{x+1}{x^2-4x+3} = \pm\frac{1}{2}$	**M1**	Equates the original curve to $\pm\frac{1}{2}$ and clears the fractions to find the boundary values. Condone sign slips.
	$x^2 - 6x + 1 = 0$, $x^2 - 2x + 5 = 0$ $x^2 - 2x + 5 = (x-1)^2 + 4 > 0$	**A1**	Both correct quadratics, with the second shown or recognised to have no real roots.
	$x^2 - 6x + 1 = 0 \Rightarrow x = \ldots$	**dM1**	Solves their three-term quadratic with real roots by any means, reaching $x = \ldots$.
	$3 - 2\sqrt{2} < x < 3 + 2\sqrt{2}$ $x = 1, 3$ satisfy the original inequality: right side zero.	**A1**	Exact strict interval read from the sketch, including $x = 1, 3$, where the original right-hand side is zero. A0 if either is excluded or extra intervals are given.

Question 13 continued

Question	Scheme	Marks	Notes
	Alternative: Squaring		
			
	Shape	**B1**	Curve from (c) with negative parts reflected: all on or above the x-axis, with a cusp on the x-axis at $x = -1$ and asymptotes $x = 1$, $x = 3$, $y = 0$.
	$4(x+1)^2 > (x^2-4x+3)^2$ $\Rightarrow \big(x^2-4x+3-2(x+1)\big)$ $\quad \times \big(x^2-4x+3+2(x+1)\big) < 0$	**M1**	Squares both sides and factorises the difference of two squares, or forms both equations $x^2-4x+3 = \pm 2(x+1)$. Condone sign slips.
	$(x^2-6x+1)(x^2-2x+5) < 0$	**A1**	oe, including both equations $x^2-6x+1=0$ and $x^2-2x+5=0$.
	$x^2-6x+1=0 \Rightarrow x = \ldots$	**dM1**	Solves their three-term quadratic by any means, reaching $x = \ldots$.
	$3-2\sqrt{2} < x < 3+2\sqrt{2}$ $x = 1$ and $x = 3$ satisfy the original inequality.	**A1**	Exact strict interval, including $x = 1, 3$, where the original right-hand side is zero. A0 if either is excluded or extra intervals are given.
		[5]	
	Total for Question 13	**14**	

Question	Scheme	Marks	Notes
14(a)			
	Shape	**B1**	Central branch with maximum on the y-axis, tending to $-\infty$ at $x = \pm 2a$; outer branches above $y = 1$, approaching it. Asymptotes shown and labelled.
	Intercepts	**B1**	$(\pm a, 0)$ and $(0, \frac{1}{4})$ marked or labelled on the central branch.
		[2]	
14(b)(i)	$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\left(\frac{x^2-a^2}{x^2-4a^2}\right) \times \frac{2x(x^2-4a^2)-2x(x^2-a^2)}{(x^2-4a^2)^2}$	**M1**	Differentiates by chain rule with quotient or product rule, or expands first; quotient of the form $\frac{vu'-uv'}{v^2}$. Condone sign slips.
	$= -\frac{12a^2x(x^2-a^2)}{(x^2-4a^2)^3}$	**A1**	Correct derivative in any form, need not be simplified.
	$x(x^2-a^2) = 0 \Rightarrow x = \ldots,\ y = \ldots$	**dM1**	Sets their derivative to zero, solves for at least one x and finds its y.
	$\left(0, \frac{1}{16}\right),\ (a, 0),\ (-a, 0)$	**A1**	All three and no others.
		[4]	
14(b)(ii)	$u = \left\lvert\frac{x^2-a^2}{x^2-4a^2}\right\rvert$ $u^2 = u \Rightarrow u = 0$ or 1	**M1**	Equates the curves and reduces to the modulus equal to 0 or 1, or the fraction equal to 0 or ± 1, by any method.
	$(\pm a, 0)$	**A1**	

Question 14 continued

Question	Scheme	Marks	Notes
	$\frac{x^2-a^2}{x^2-4a^2} = -1 \Rightarrow 2x^2 = 5a^2$ $\left(\pm a\sqrt{\frac{5}{2}},\ 1\right)$	**A1**	Both points, oe such as $\pm\frac{\sqrt{10}}{2}a$ or awrt $\pm 1.58a$, and no others. Rejecting fraction $= 1$ need not be shown.
		[3]	
14(c)			
	Shape	**B1**	C_2: on or above the x-axis, touching at $(\pm a, 0)$, maximum on the y-axis, asymptotes $x = \pm 2a$ and $y = 1$, outer branches above $y = 1$.
	Shape	**B1**	C_3: C_1 with parts below the x-axis reflected, cusps at $(\pm a, 0)$, same asymptotes.
	Labelled point	**B1**	Curves labelled, meeting at $(\pm a, 0)$ and crossing on $y = 1$; C_2 below C_3 between the crossings, above outside them.
	Critical values $x = \pm a\sqrt{\frac{5}{2}}$	**M1**	Uses their x-coordinates of the intersections on $y = 1$ as the ends of the interval.
	$-a\sqrt{\frac{5}{2}} \leq x \leq a\sqrt{\frac{5}{2}}$	**A1**	oe, including set notation. A0 for strict inequalities or extra intervals.
		[5]	
	Total for Question 14	**14**	