CORE PURE / MISCELLANEOUS PURE

Hyperbolic Functions

Mark scheme 18 questions · 159 marks

EXAM BOARDS Edexcel 9FM0 · AQA 7367 · OCR A H245 · OCR MEI H645 · CIE 9231

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	Total	**159**	

Reading the mark scheme

These rules apply throughout, unless a note says otherwise.

Marks

Every mark is worth one, so **M1 A1** on a row is two marks. Only marks coded ft follow through.

Code	Meaning
M1	Method: a correct method, applied to this question. A slip doesn't cost it if the method is clear. A formula must be used, not just quoted.
dM1	Dependent method: as M1, but only if the method mark before it was earned, unless its note names another.
A1	Accuracy: a correct answer or step. It needs the method mark it follows.
A1ft	As A1, following through from an earlier answer that the note names.
A1 AG	The last mark of a part whose answer is printed in the question: complete working, with no errors.
B1	A correct value, fact or statement. It needs no method mark.
B1ft	As B1, following through from an earlier answer that the note names.
E1	A written explanation, interpretation or conclusion. Concluding a proof by induction needs every earlier mark.

Abbreviations

ft and AG qualify a code in the Marks column. The others appear in the notes. A grey line gives the rule where the name alone doesn't.

ft	follow through Correct work from the student's own earlier wrong value earns the mark.	**AG**	answer given The answer is printed in the question, so every step to it must be shown.
cao	correct answer only No follow-through from an earlier error.	**cso**	correct solution only No error anywhere in the part, not just in its answer.
oe	or equivalent	**awrt**	answers which round to
soi	seen or implied Earned if written, or if later working shows it was used.	**isw**	ignore subsequent working A correct answer, once reached, isn't spoiled by later working.
SC	special case Credit, as the note says, for work outside the main method.	**their**	the student's own earlier value
condone	a slip that doesn't cost the mark	[ ]	working the mark doesn't need
s.f.	significant figures	**d.p.**	decimal places

Reading the scheme

- Each row is a line of working, with the marks it earns beside it. Where a row carries two marks, each note starts with its code.
- [$\boldsymbol{n}$] ends each part with its total.
- A shaded *Alternative* is a second method for the same total. Mark an answer by the method it follows.
- Any other valid method earns the equivalent marks, unless the question names a method or says *hence*.
- A sketch earns one B1 for each feature named below its figure.

Accuracy and answers

- Unless a question says otherwise, give answers to 3 significant figures, and angles in degrees to 1 decimal place. Angles in radians take 3 significant figures.
- A final answer with more figures is accepted if it rounds to the scheme's value; one with fewer is not. An answer spoiled by rounding too early loses its accuracy mark.
- Where a question asks for an exact answer, a decimal loses the final accuracy mark. Otherwise either is accepted.
- Wrong or missing units cost nothing unless the question asks for them.
- An omitted $\mathrm{d}\theta$, $\mathrm{d}x$ or $\mathrm{d}u$ in an integral is not penalised where the variable of integration is clear from the integrand and the limits. It is penalised where it is genuinely ambiguous: a substitution part-way through, two variables in play, or limits in the wrong variable.
- A correct answer with no working earns full marks, unless the question needs the working: show that, prove, hence, a method it names, or an exact answer. A wrong answer with no working earns nothing. Calculator answers are treated the same way.
- A correct answer, once reached, isn't spoiled by later working (isw).

Value of g

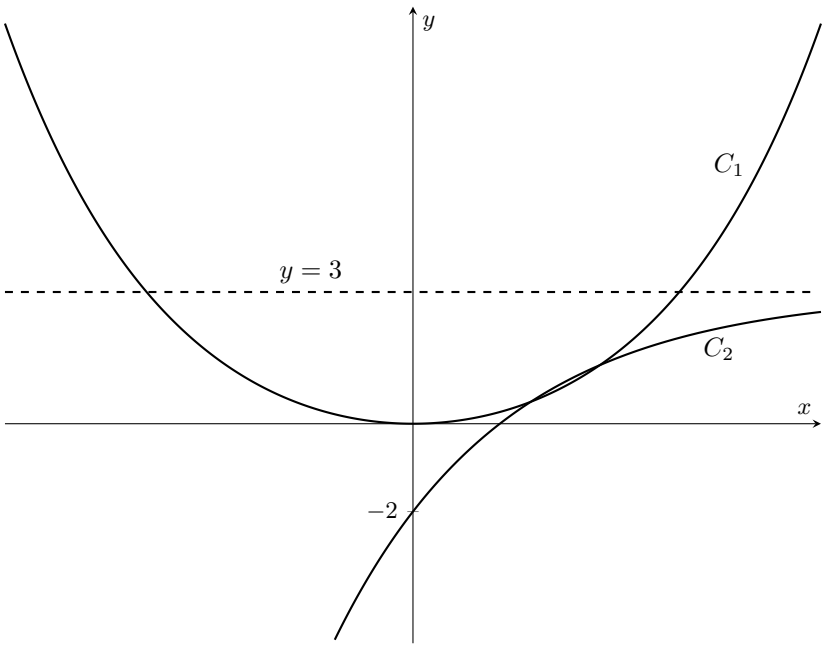
- Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question states a value, which is then the only one accepted. Answers that use $g = 9.8$ may be given to 2 or 3 significant figures.
- Where a question doesn't state g, an answer worked with $g = 10$ throughout earns the same marks, keeping any value printed in the question. The worked solutions list the final answers with $g = 10$.

Question	Scheme	Marks	Notes
1	$2\left(\frac{e^x-e^{-x}}{2}\right)+5\left(\frac{e^x+e^{-x}}{2}\right)=6$ $\Rightarrow \ldots e^{2x}+\ldots e^x+\ldots=0$	**M1**	Substitutes both exponential forms and multiplies through by a multiple of e^x to reach a three-term quadratic in e^x. Condone sign and coefficient slips.
	$7e^{2x}-12e^x+3=0$	**A1**	oe, any multiple or arrangement, such as $7e^{2x}+3=12e^x$.
	$e^x=\frac{6\pm\sqrt{15}}{7}\Rightarrow x=\ln(\ldots)$	**M1**	Solves their three-term quadratic in e^x by formula, completing the square, factorising or calculator, and takes ln of at least one positive root.
	$x=\ln\left(\frac{6+\sqrt{15}}{7}\right),\ x=\ln\left(\frac{6-\sqrt{15}}{7}\right)$	**A1**	cso. Both, in any exact log form such as $\ln(6\pm\sqrt{15})-\ln 7$. A0 if either is rejected.
	Total for Question 1	**4**	

Question	Scheme	Marks	Notes
2(a)	$\tanh x = \dfrac{\frac{1}{2}(e^x - e^{-x})}{\frac{1}{2}(e^x + e^{-x})}$ $= \dfrac{(e^x - e^{-x})e^x}{(e^x + e^{-x})e^x}$	**M1**	Writes $\sinh x / \cosh x$ using both exponential definitions and multiplies numerator and denominator by e^x; or divides the printed fraction through by e^x.
	$= \dfrac{e^{2x} - 1}{e^{2x} + 1}$	**A1 AG**	Correct definitions leading without error to the printed result; combined steps accepted.
		[2]	
2(b)	$\dfrac{e^{2(x-\ln 2)} - 1}{e^{2(x-\ln 2)} + 1} = \dfrac{3}{7}$ $\Rightarrow e^{2(x-\ln 2)} = \ldots$	**M1**	Substitutes $x - \ln 2$, or a defined variable for it, into the result of (a), sets it equal to $\frac{3}{7}$ and rearranges for the exponential. Condone sign slips.
	$e^{2(x-\ln 2)} = \frac{5}{2}$	**A1**	oe, such as $e^{2x} = 10$, or $e^{2y} = \frac{5}{2}$ with $y = x - \ln 2$.
	$2(x - \ln 2) = \ln$ "$\frac{5}{2}$"	**M1**	Takes ln of both sides of their $e^{2(x-\ln 2)} = c$, $c > 0$, giving a linear equation in x.
	$2x = \ln$ "$\frac{5}{2}$" $+ \ln 4 = \ln(\ldots)$	**dM1**	Uses $2\ln 2 = \ln 4$ and $\ln a + \ln b = \ln ab$ to reach a single log; or $e^{-2\ln 2} = \frac{1}{4}$ before taking logs.
	$x = \frac{1}{2}\ln 10$	**A1**	cso. $x = \frac{1}{2}\ln 10$ or $k = 10$; $\ln\sqrt{10}$ or a decimal scores A0.
		[5]	
Total for Question 2		**7**	

Question	Scheme	Marks	Notes
3(a)	$\cosh x + \sinh x = \frac{(e^x+e^{-x})+(e^x-e^{-x})}{2}$	**M1**	Substitutes both exponential definitions into either expression and combines over 2. Condone sign slips.
	$= \frac{2e^x}{2} = e^x$	**A1**	Correct combined numerator simplified to e^x without error.
	$\cosh x - \sinh x = \frac{(e^x+e^{-x})-(e^x-e^{-x})}{2}$ $= \frac{2e^{-x}}{2} = e^{-x}$	**A1 AG**	Correct substituted expression with the subtraction applied to both terms of $\sinh x$, leading without error to e^{-x}.
		[3]	
3(b)	$2(e^x + e^{-x}) - \frac{3}{2}(e^x - e^{-x}) = 5$ $\Rightarrow \ldots e^{2x} + \ldots e^x + \ldots = 0$	**M1**	Substitutes exponential forms, or uses (a), and multiplies by e^x to reach a three-term quadratic in e^x. Condone sign and coefficient slips.
	$e^{2x} - 10e^x + 7 = 0$	**A1**	oe, any multiple, terms in any order, or in a substituted letter for e^x.
	$e^x = \frac{10 \pm \sqrt{72}}{2} \Rightarrow x = \ln(\ldots)$	**M1**	Solves their three-term quadratic in e^x by any means, calculator allowed, and takes ln of at least one positive root.
	$x = \ln(5 + 3\sqrt{2})$ or $x = \ln(5 - 3\sqrt{2})$	**A1**	Both exact, and no others. Accept equivalent exact forms, such as $\ln \frac{10 \pm \sqrt{72}}{2}$, or $\ln 7 - \ln(5 + 3\sqrt{2})$ for the second.
		[4]	
	Total for Question 3	**7**	

Question	Scheme	Marks	Notes
4(a)	$\operatorname{sech} 2x = \dfrac{1}{2\cosh^2 x - 1}$ $= \dfrac{1}{\frac{2}{\operatorname{sech}^2 x} - 1}$	**M1**	Takes the reciprocal of the identity and replaces $\cosh^2 x$ by $\frac{1}{\operatorname{sech}^2 x}$, or substitutes $\operatorname{sech}^2 x = \frac{1}{\cosh^2 x}$ into the RHS.
	$= \dfrac{\operatorname{sech}^2 x}{2 - \operatorname{sech}^2 x}$	**A1 AG**	Printed result reached without error from a correct substituted expression; combined simplification accepted. Working from the RHS must end at $\frac{1}{\cosh 2x}$ via the identity.
		[2]	
4(b)(i)	$0 < f(x) < 1$	**B1**	oe, such as $(0, 1)$. B0 if either bound is included.
		[1]	
4(b)(ii)	$\operatorname{sech} 2x = \dfrac{\operatorname{sech}^2 x}{2 - \operatorname{sech}^2 x}$ $0 < \operatorname{sech} x < 1 \Rightarrow 2 - \operatorname{sech}^2 x > 1$	**M1**	Uses (a) with $0 < \operatorname{sech} x < 1$ to bound the denominator or the fraction, such as $2 - \operatorname{sech}^2 x > 1$.
	$\Rightarrow \operatorname{sech} 2x < \operatorname{sech}^2 x$	**A1**	
	$0 < \operatorname{sech} x < 1 \Rightarrow \operatorname{sech}^2 x < \operatorname{sech} x$ $\Rightarrow \operatorname{sech} 2x < \operatorname{sech} x$	**A1 AG**	Justifies $\operatorname{sech}^2 x < \operatorname{sech} x$ from $0 < \operatorname{sech} x < 1$ and concludes the printed inequality. A0 if that justification is missing.
		[3]	
Total for Question 4		**6**	

Question	Scheme	Marks	Notes
5(a)			
	Shape	**B1**	C_1: U shape symmetric about the y-axis, minimum at the origin, meeting the axes only there.
	Shape	**B1**	C_2: increasing curve, approaching its asymptote from below as $x \to \infty$, unbounded below as $x \to -\infty$.
	Asymptotes	**B1**	$y = 3$ stated as the asymptote of C_2; none drawn for C_1.
	Intercepts	**B1**	$(0, -2)$ and $\left(\frac{1}{2}\ln\frac{5}{3}, 0\right)$ oe, or awrt 0.255 for the x-intercept.
		[4]	
5(b)	$e^{2x} + e^{-2x} - 2 = 3 - 5e^{-2x}$	**M1**	Replaces $\cosh 2x$ by $\frac{1}{2}(e^{2x} + e^{-2x})$ and multiplies through to reach a three-term quadratic in e^{2x}. Condone sign and coefficient slips.
	$e^{4x} - 5e^{2x} + 6 = 0$	**A1**	oe, in e^{2x} or a substituted variable.
	$(e^{2x} - 2)(e^{2x} - 3) = 0$ $\Rightarrow 2x = \ln\ldots$	**M1**	Solves their three-term quadratic by any means, calculator allowed, and takes logs of a positive root to find x.

Question 5 continued

Question	Scheme	Marks	Notes
	$x = \frac{1}{2}\ln 2,\ x = \frac{1}{2}\ln 3$	**A1**	Both, in the form $\frac{1}{2}\ln k$ only, and no others. A0 for decimals or $\ln\sqrt{2}$, $\ln\sqrt{3}$.
		[4]	
	Total for Question 5	**8**	

Question	Scheme	Marks	Notes
6(a)	$\cosh 2x = \frac{e^{2x}+e^{-2x}}{2}$ $2\sinh^2 x = 2\left(\frac{e^x-e^{-x}}{2}\right)^2$ $= \frac{e^{2x}-2+e^{-2x}}{2}$	**M1**	Writes $\cosh 2x$ in exponentials and expands the square of the exponential form of $\sinh x$. Either side may be started from.
	$\cosh 2x = 1 + \frac{e^{2x}-2+e^{-2x}}{2}$ $= 1 + 2\sinh^2 x$	**A1 AG**	Correct exponential forms leading without error to the printed identity. If both sides are worked separately, needs a closing statement such as LHS = RHS.
		[2]	
6(b)	$2(1+2\sinh^2 x) - 7\sinh x = 4$ $\Rightarrow 4\sinh^2 x - 7\sinh x - 2 = 0$ $\Rightarrow \sinh x = \ldots$	**M1**	Substitutes (a) to form a three-term quadratic in $\sinh x$ and solves it by any method. Condone sign slips.
	$\sinh x = 2,\ \sinh x = -\frac{1}{4}$	**A1**	
	$x = \ln\left(\text{“2”} + \sqrt{\text{“2”}^2 + 1}\right)$	**M1**	Uses $\operatorname{arsinh} u = \ln\left(u + \sqrt{u^2+1}\right)$, or solves a quadratic in e^x, for at least one of their values of $\sinh x$. Not for decimals alone.
	$x = \ln(2+\sqrt{5}),\ x = \ln\left(\frac{\sqrt{17}-1}{4}\right)$	**A1**	Both, in any equivalent logarithmic form, and no others. Decimals alone score A0.
		[4]	
Total for Question 6		**6**	

Question	Scheme	Marks	Notes
7(a)	$x = \cosh y = \frac{e^y + e^{-y}}{2}$ $\Rightarrow e^{2y} - 2xe^y + 1 = 0$	**M1 A1**	**M1** Replaces $\cosh y$ by exponentials and multiplies through to form a three-term quadratic in e^y or a substituted letter. Condone sign and coefficient slips. **A1** Any equivalent arrangement, in e^y or another letter.
	$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$	**M1**	Solves their three-term quadratic in e^y by the formula or completing the square, reaching $e^y = \ldots$. Condone slips.
	$e^y = x \pm \sqrt{x^2 - 1}$	**A1**	Both roots.
	$y \geqslant 0 \Rightarrow e^y \geqslant 1$ $x - \sqrt{x^2 - 1} = \frac{1}{x + \sqrt{x^2 - 1}} \leqslant 1$ $\Rightarrow \operatorname{arcosh} x = \ln\left(x + \sqrt{x^2 - 1}\right)$	**A1 AG**	Rejects the minus root using $y \geqslant 0$, with a reason it is at most 1 (e.g. product of roots is 1), then takes logs. A0 for an unexplained rejection.
		[5]	
7(b)	$x = \sinh y,\ x + 2 = \cosh y$ $\Rightarrow \cosh y - \sinh y = 2$	**M1**	Writes both curves in terms of $\sinh y$ and $\cosh y$ at a common point and eliminates x. Condone sign slips.
	$\cosh y - \sinh y = e^{-y} \Rightarrow e^{-y} = 2$	**A1**	oe, such as $y = -\ln 2$.
	$y = \operatorname{arcosh}(x + 2) \geqslant 0 \Rightarrow e^{-y} \leqslant 1$ Contradiction, so the graphs do not intersect	**A1 AG**	Uses $y \geqslant 0$ from the range of arcosh to contradict $e^{-y} = 2$ oe, and concludes no intersection. A0 if the conclusion is missing.
		[3]	
	Total for Question 7	**8**	

Question	Scheme	Marks	Notes
8(a)	$\cosh k - 2\sinh k > 0$ $\frac{e^k+e^{-k}}{2} - (e^k - e^{-k}) > 0$	**M1**	States the argument must be positive and writes $\cosh k - 2\sinh k$ in exponentials, or uses $\tanh k < \frac{1}{2}$. Condone sign slips.
	$\Rightarrow 3e^{-k} > e^k \Rightarrow e^{2k} < 3$ $\Rightarrow k < \frac{1}{2}\ln 3$	**A1 AG**	Correct inequality rearranged without error to $k < \frac{1}{2}\ln 3$; via $\tanh k < \frac{1}{2}$ needs the logarithmic form of artanh.
		[2]	
8(b)(i)	$f'(x) = \frac{\sinh x - 2\cosh x}{\cosh x - 2\sinh x}$	**M1 A1**	**M1** Obtains the form $\frac{p\sinh x + q\cosh x}{\cosh x - 2\sinh x}$, $p, q \neq 0$. Condone sign slips. **A1** oe
		[2]	
8(b)(ii)	$f''(x) =$ $\frac{(\cosh x - 2\sinh x)^2 - (\sinh x - 2\cosh x)^2}{(\cosh x - 2\sinh x)^2}$	**M1 A1**	**M1** Applies the quotient or product rule to their $f'(x)$ of the form $\frac{p\sinh x + q\cosh x}{\cosh x - 2\sinh x}$. Condone sign slips. **A1** Correct, oe, unsimplified.
	Numerator $= 3\sinh^2 x - 3\cosh^2 x$ $= -3 \Rightarrow a = -3$	**A1**	cso. Numerator expanded and simplified using $\cosh^2 x - \sinh^2 x = 1$ or exponentials; $a = -3$ stated or the form with -3 shown.
		[3]	
8(c)	$f(0) = 0,\ f'(0) = -2,\ f''(0) = -3$	**B1**	
	$f(x) \approx 0 + (-2)x + \frac{-3}{2!}x^2$	**M1**	Substitutes their three values into $f(0) + xf'(0) + \frac{x^2}{2!}f''(0)$.
	$f(x) \approx -2x - \frac{3}{2}x^2$	**A1**	
		[3]	
8(d)	$-2(-0.1) - \frac{3}{2}(-0.1)^2 = 0.185$ $f(-0.1) = 0.18676\ldots$ $\frac{0.18676 - 0.185}{0.18676} \times 100$	**M1**	Evaluates their quadratic and $f(x)$ at $x = -0.1$, and divides the difference by the exact value, times 100.
	0.942%	**A1**	awrt 0.942%; condone a negative sign.
		[2]	
	Total for Question 8	**12**	

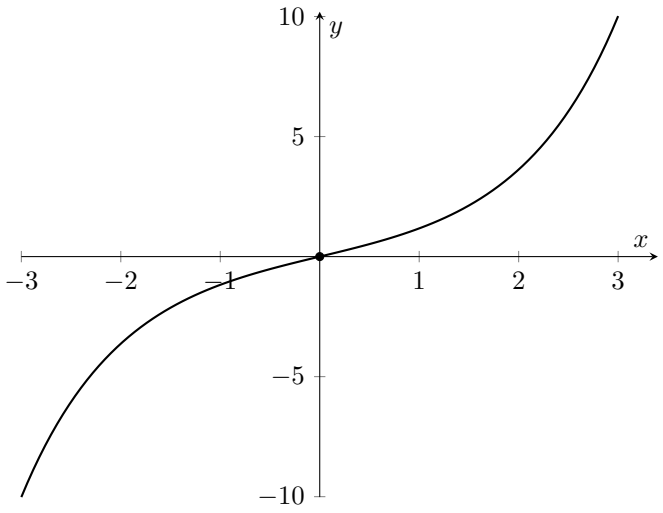
Question	Scheme	Marks	Notes
9(a)	$R\cosh\alpha = 7,\ R\sinh\alpha = 5$ $\Rightarrow R^2 = 7^2 - 5^2$	**M1**	Compares coefficients to obtain $R\cosh\alpha$ and $R\sinh\alpha$, and uses $\cosh^2\alpha - \sinh^2\alpha = 1$ to find R^2.
	$R = 2\sqrt{6}$	**A1**	oe, or awrt 4.90.
	$\tanh\alpha = \frac{5}{7} \Rightarrow \alpha = \text{artanh}\,\frac{5}{7} = \ldots$	**M1**	Obtains $\tanh\alpha$, or $\cosh\alpha$ or $\sinh\alpha$ with their R, and applies the inverse function or its log form to find α.
	$\alpha = 0.896$	**A1**	awrt 0.896; $\frac{1}{2}\ln 6$ alone does not score.
		[4]	
9(b)	$\cosh(x - \text{“}0.896\text{”}) = \dfrac{10}{\text{“}2\sqrt{6}\text{”}}$ $\Rightarrow x = \text{“}0.896\text{”} \pm \text{arcosh}\,\frac{5}{\sqrt{6}} = \ldots$	**M1**	Divides by their R and takes arcosh, or its log form, of their $\frac{10}{R} > 1$, adding their α. Condone a missing $\pm$.
	$x = 2.24$	**A1**	awrt 2.24.
	$x = -0.44$	**A1**	awrt -0.44, with no further solutions.
		[3]	
9(c)	$\frac{7}{2}(e^x + e^{-x}) - \frac{5}{2}(e^x - e^{-x}) = 10$ $\Rightarrow e^{2x} - 10e^x + 6 = 0$	**M1 A1**	**M1** Replaces both functions by exponential forms and multiplies by e^x to reach a three-term quadratic in e^x. Condone sign and coefficient slips. **A1** oe.
	$e^x = \ldots \Rightarrow x = \ln(\ldots)$	**M1**	Solves their three-term quadratic in e^x by any means and takes ln of at least one positive root.
	$x = \ln(5 \pm \sqrt{19})$ $[= 2.24,\ -0.44]$	**A1**	Both, exact or awrt 2.24 and -0.44, with no others.
		[4]	
Total for Question 9		**11**	

Question	Scheme	Marks	Notes
10	$\tanh t = \frac{e^t - e^{-t}}{e^t + e^{-t}} = \frac{e^{2t}-1}{e^{2t}+1}$	**M1**	Writes tanh of x, y or $x+y$ in exponentials, as $\frac{e^t - e^{-t}}{e^t + e^{-t}}$ or $\frac{e^{2t}-1}{e^{2t}+1}$. Not for a definition quoted alone.
	$a = e^{2x},\ b = e^{2y}$: $\tanh x = \frac{a-1}{a+1},\ \tanh y = \frac{b-1}{b+1}$ $\tanh(x+y) = \frac{ab-1}{ab+1}$	**A1**	All three correct in any exponential form; new letters need not be introduced.
	$\text{RHS} = \frac{(a-1)(b+1)+(b-1)(a+1)}{(a+1)(b+1)+(a-1)(b-1)}$	**M1**	Substitutes their exponential forms of $\tanh x$ and $\tanh y$ into the RHS and clears the fractions within it. Condone slips.
	$= \frac{2ab-2}{2ab+2}$	**dM1**	Expands both numerator and denominator. Condone sign slips; need not be simplified.
	$= \frac{ab-1}{ab+1} = \tanh(x+y)$	**A1 AG**	cso: correct combined fraction simplified without error to the form of $\tanh(x+y)$, with a conclusion, minimum RHS = LHS.
	Total for Question 10	**5**	

Question	Scheme	Marks	Notes
11(a)	$\tanh x = \frac{e^{2x}-1}{e^{2x}+1}$, $\tanh 3x = \frac{e^{6x}-1}{e^{6x}+1}$	**M1**	Writes $\tanh x$ and $\tanh 3x$ as quotients of exponentials, in e^x or e^{2x}. Condone sign slips.
	$y = e^{2x}$: $\tanh x = \frac{y-1}{y+1}$, $\tanh 3x = \frac{y^3-1}{y^3+1}$	**A1**	Both correct in one variable, such as $y = e^{2x}$ or e^x.
	$\frac{3\left(\frac{y-1}{y+1}\right)+\left(\frac{y-1}{y+1}\right)^3}{1+3\left(\frac{y-1}{y+1}\right)^2}$ $= \frac{3(y-1)(y+1)^2+(y-1)^3}{(y+1)((y+1)^2+3(y-1)^2)}$	**M1**	Substitutes their exponential form of $\tanh x$ into the RHS and clears the inner fractions to a single fraction. Condone slips.
	$= \frac{(y-1)\cdot 4(y^2+y+1)}{(y+1)\cdot 4(y^2-y+1)}$	**dM1**	Expands and simplifies numerator and denominator, by factorising or full expansion, to a form comparable with their $\tanh 3x$.
	$= \frac{y^3-1}{y^3+1} = \tanh 3x$	**A1 AG**	Correct substituted fraction leading without error to $\frac{y^3-1}{y^3+1}$, identified as $\tanh 3x$ or LHS = RHS. Combined steps accepted.
		[5]	
11(b)	$\frac{3t+t^3}{1+3t^2} = 2t \Rightarrow t - 5t^3 = 0 \Rightarrow t = \ldots$	**M1**	Equates the expression from (a) to $2\tanh x$, clears the fraction and solves the cubic by any means. Condone sign slips.
	$\tanh x = 0$, $\tanh x = \pm\frac{1}{\sqrt{5}}$	**A1**	All three values.
	$x = \pm\frac{1}{2}\ln\left(\frac{1+\frac{1}{\sqrt{5}}}{1-\frac{1}{\sqrt{5}}}\right)$	**M1**	Uses $\operatorname{artanh} u = \frac{1}{2}\ln\frac{1+u}{1-u}$, or the exponential definition of $\tanh x$, with their nonzero value, $\lvert u\rvert < 1$.
	$x = 0$, $x = \pm\ln\left(\frac{1+\sqrt{5}}{2}\right)$	**A1**	All three, no others. Accept single logs with rationalised surds, such as $\pm\frac{1}{2}\ln\frac{3+\sqrt{5}}{2}$ or $\ln\frac{\sqrt{5}-1}{2}$; decimals score A0.
		[4]	
Total for Question 11		**9**	

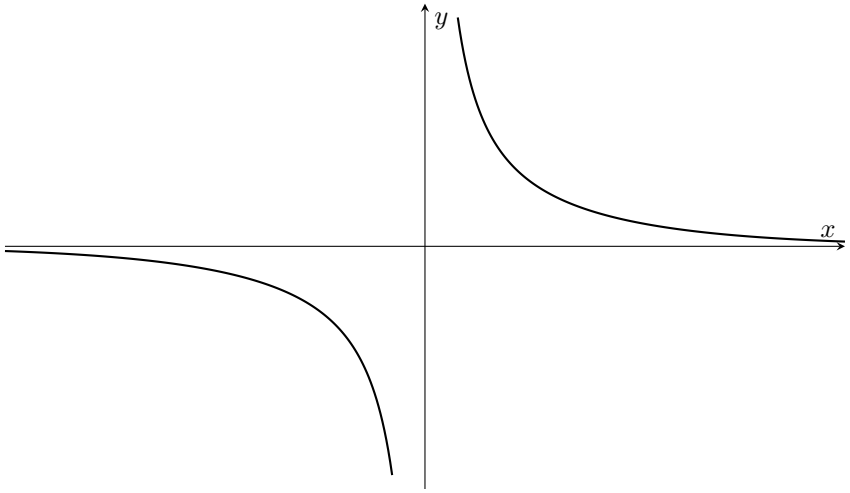
Question	Scheme	Marks	Notes
12(a)(i)	$1+2\sinh t=0 \Rightarrow \sinh t=-\frac{1}{2}$	**M1 A1**	**M1** Sets $1+2\sinh t=0$.
	$t=\ln\left(-\frac{1}{2}+\sqrt{\frac{1}{4}+1}\right)$	**M1**	Substitutes their $\sinh t$ into $\ln\left(u+\sqrt{u^2+1}\right)$, or solves a quadratic in e^t and takes logarithms. Not for $\text{arsinh}\left(-\frac{1}{2}\right)$ alone.
	$t=\ln\left(\frac{\sqrt{5}-1}{2}\right)$	**A1**	Any exact single logarithm, oe. A0 for a decimal, or if a logarithm of the negative root is retained.
		[4]	
12(a)(ii)	$\cosh t=\sqrt{1+\frac{1}{4}}$ $[=\frac{\sqrt{5}}{2}]$ $y=3\left(\text{“}\frac{\sqrt{5}}{2}\text{”}\right)-\frac{1}{2}$	**M1**	Obtains $\cosh t$ from their $\sinh t$ or their t by any valid means, and substitutes both into y.
	$y=2.85$	**A1**	awrt 2.85.
		[2]	
12(b)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3\sinh t+\cosh t}{2\cosh t}$	**M1 A1**	**M1** Divides their $\frac{\mathrm{d}y}{\mathrm{d}t}$ by their $\frac{\mathrm{d}x}{\mathrm{d}t}$, each differentiated to hyperbolic terms. Condone sign slips.
	$\sinh(\ln 2)=\frac{3}{4},\ \cosh(\ln 2)=\frac{5}{4}$ $B\left(\frac{5}{2},\frac{9}{2}\right)$	**B1**	Both coordinates of B correct; the hyperbolic values may be implied by them.
	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3\left(\frac{3}{4}\right)+\frac{5}{4}}{2\left(\frac{5}{4}\right)}=\frac{7}{5}$	**A1**	oe, such as 1.4.
	$y-\text{“}\frac{9}{2}\text{”}=\text{“}\frac{7}{5}\text{”}\left(x-\text{“}\frac{5}{2}\text{”}\right)$	**M1**	Forms a straight line through their B with their gradient at $t=\ln 2$, by any form of line equation.
	$y=\frac{7}{5}x+1$	**A1**	Any correct form, such as $7x-5y+5=0$.
		[6]	
	Total for Question 12	**12**	

Question	Scheme	Marks	Notes
13(a)	$\tanh q = \frac{e^q - e^{-q}}{e^q + e^{-q}} = \frac{e^{2q}-1}{e^{2q}+1}$	**M1 A1**	**M1** Uses the exponential definition of $\tanh q$ and multiplies numerator and denominator by e^q (or e^{-q}). Condone a sign slip.
	$e^{2q} = \frac{1+t}{1-t}$ $\tanh q = \frac{\frac{1+t}{1-t}-1}{\frac{1+t}{1-t}+1} = \ldots$	**M1**	Obtains e^{2q} or e^q in terms of t, substitutes into their exponential form and attempts to combine the fractions.
	$= \frac{(1+t)-(1-t)}{(1+t)+(1-t)} = t$	**A1 AG**	Correct substituted expression simplified without error to t; combined steps accepted.
		[4]	
13(b)	$6(1-\tanh^2 x) = 5 - \tanh x$ $\Rightarrow 6\tanh^2 x - \tanh x - 1 = 0 \Rightarrow \tanh x = \ldots$	**M1**	Uses $\operatorname{sech}^2 x = 1 - \tanh^2 x$ to form a three-term quadratic in $\tanh x$ and solves it by any means. Condone a sign slip in the identity.
	$\tanh x = \frac{1}{2},\ \tanh x = -\frac{1}{3}$	**A1**	
	$x = \frac{1}{2}\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right),\ \frac{1}{2}\ln\left(\frac{1-\frac{1}{3}}{1+\frac{1}{3}}\right)$	**M1**	Uses (a), the logarithmic form of artanh, or exponentials to find x for at least one of their values in $(-1, 1)$.
	$x = \frac{1}{2}\ln 3,\ x = -\frac{1}{2}\ln 2$	**A1**	Both, in any logarithmic form, e.g. $\frac{1}{2}\ln\frac{1}{2}$ or $\ln\sqrt{3}$; decimals alone score A0.
		[4]	
	Total for Question 13	**8**	

Question	Scheme	Marks	Notes
14(a)			
	Shape	**B1**	Increasing curve in the first and third quadrants, steepening as $\|x\|$ grows.
	Symmetry	**B1**	Through O, with rotational symmetry about O.
		[2]	
14(b)	$u = \frac{e^x - e^{-x}}{2} \Rightarrow 2ue^x = e^{2x} - 1$	**M1**	Uses the exponential definition and multiplies by e^x to form a quadratic in e^x. Condone sign slips.
	$e^{2x} - 2ue^x - 1 = 0$	**A1**	oe, in any letter for e^x.
	$e^x = \frac{2u \pm \sqrt{4u^2+4}}{2}$	**M1**	Solves their three-term quadratic in e^x by the formula or completing the square. Condone slips.
	$e^x = u \pm \sqrt{u^2+1}$	**A1**	
	$u - \sqrt{u^2+1} < 0$ and $e^x > 0$ $\Rightarrow x = \ln\left(u + \sqrt{u^2+1}\right)$	**A1 AG**	Rejects the minus root by stating it is negative while $e^x > 0$, then takes logs.
		[5]	
14(c)(i)	$9(1 + \sinh^2 x) - 24\sinh x = -7$	**M1**	Replaces $\cosh^2 x$ using $\cosh^2 x - \sinh^2 x = 1$. Condone a sign slip in the identity.
	$9\sinh^2 x - 24\sinh x + 16 = 0$	**A1 AG**	Correct substituted equation leading without error to the printed equation.
		[2]	

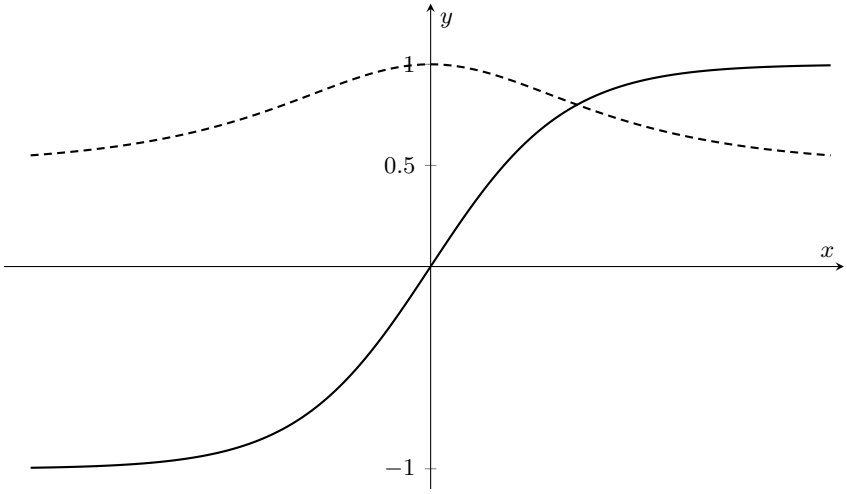
Question 14 continued

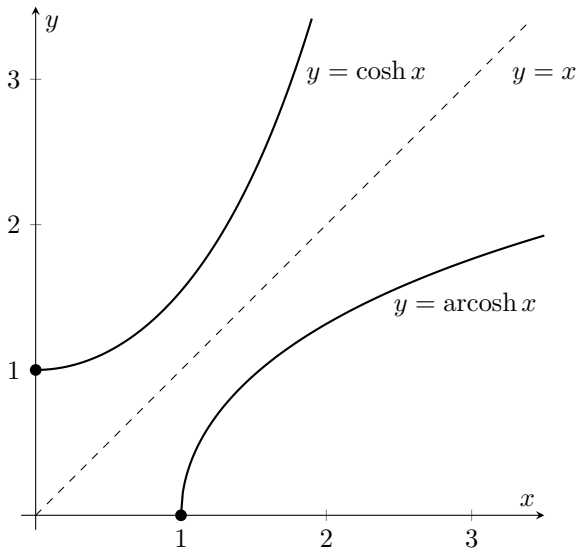
Question	Scheme	Marks	Notes
14(c)(ii)	$(3\sinh x - 4)^2 = 0 \Rightarrow \sinh x = \ldots$	**M1**	Solves the quadratic in $\sinh x$ by any method, leading to $\sinh x = \ldots$.
	$\sinh x = \frac{4}{3}$ (repeated)	**A1**	Shown to be the only root, by a squared factor or zero discriminant.
	$\sinh x$ is one-to-one, so one real x	**E1**	States that $\sinh x$ is one-to-one, increasing or takes each value once oe, for their single value. May cite the graph in (a).
	$x = \ln\left("\frac{4}{3}" + \sqrt{"\frac{16}{9}" + 1}\right)$	**M1**	Substitutes their value of $\sinh x$ into the result of (b), or solves the resulting quadratic in e^x.
	$x = \ln 3$	**A1**	Must be $\ln 3$; a decimal such as 1.10 does not score.
		[5]	
Total for Question 14		**14**	

Question	Scheme	Marks	Notes
15(a)			
	Shape	**B1**	Two branches, in the first and third quadrants, each approaching both axes without crossing either.
	Asymptotes	**B1**	$x = 0$ stated; a drawn line alone does not score.
	Asymptotes	**B1**	$y = 0$ stated; a drawn line alone does not score.
		[3]	
15(b)	$\coth^2 x - \operatorname{cosech}^2 x = \frac{\cosh^2 x - 1}{\sinh^2 x}$ $\cosh^2 x - \sinh^2 x$ $= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4}$	**M1**	Expresses both terms by the exponential definitions, directly or via $\cosh x$ and $\sinh x$, over a common denominator. Condone sign slips.
	$= \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{4} = 1$	**A1**	Correct expansion of the squared exponential brackets, oe in a direct route. Not for quoting $\cosh^2 x - \sinh^2 x = 1$.
	$\coth^2 x - \operatorname{cosech}^2 x = \frac{\sinh^2 x}{\sinh^2 x} = 1$	**A1 AG**	Printed identity reached from correct exponential working without error, with a concluding statement.
		[3]	
15(c)	$\coth^2 x - 1 = 5 - \coth x$ $\Rightarrow \coth^2 x + \coth x - 6 = 0 \Rightarrow \coth x = \ldots$	**M1**	Uses (b) to form a three-term quadratic in $\coth x$, or an equivalent quadratic such as in e^{2x}, and solves it by any means. Condone a sign slip in the identity.
	$\coth x = 2,\ \coth x = -3$	**A1**	

Question 15 continued

Question	Scheme	Marks	Notes
	$\tanh x = \text{“}\frac{1}{2}\text{”} \Rightarrow x = \frac{1}{2}\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right)$	**M1**	Uses, for at least one of their values, $\tanh x = \frac{1}{\coth x}$ with the artanh log form, an arcoth log form, or exponentials, to reach $x = \ldots$ in logs.
	$x = \frac{1}{2}\ln 3,\ x = -\frac{1}{2}\ln 2$	**A1**	Both, in any exact log form such as $\ln\sqrt{3}$ or $\frac{1}{2}\ln\frac{1}{2}$, and no others.
		[4]	
	Total for Question 15	**10**	

Question	Scheme	Marks	Notes
16(a)			
	Shape	**B1**	$y = \tanh x$: increasing through O, approaching $y = \pm 1$, with both asymptotes shown or labelled.
	Turning points	**B1**	Second curve symmetric in the y-axis, with maximum at $(0, 1)$ labelled or indicated.
	Asymptotes	**B1**	Second curve approaches $y = \frac{1}{2}$ in both directions, staying above it.
	Curves meet exactly once	**E1**	States exactly one intersection, from a sketch showing both curves crossing once, for $x > 0$.
		[4]	
16(b)	$\frac{\sinh x}{\cosh x} = \frac{1}{2}\left(1 + \frac{1}{\cosh x}\right)$ $\Rightarrow e^x - e^{-x} = \frac{1}{2}\left(e^x + e^{-x}\right) + 1$	**M1**	Replaces $\tanh x$ and $\operatorname{sech} x$ by exponentials, directly or via sinh and cosh, and multiplies through towards a quadratic in e^x. Condone sign and coefficient slips.
	$e^{2x} - 2e^x - 3 = 0$	**A1**	oe, with all terms collected.
	$(e^x - 3)(e^x + 1) = 0 \Rightarrow e^x = 3$ $(e^x \neq -1)$	**M1**	Solves their three-term quadratic in e^x by any means and rejects the negative root.
	$x = \ln 3$	**A1**	Must be in the form $\ln 3$; 1.10 alone scores A0.
		[4]	
	Total for Question 16	**8**	

Question	Scheme	Marks	Notes
17(i)			
	Shape	**B1**	Increasing convex curve starting at $(0, 1)$, drawn for $x \geq 0$ only, labelled.
	Turning points	**B1**	States the gradient at $x = 0$ is 0.
	Shape	**B1**	Increasing concave curve starting at $(1, 0)$, recognisably the reflection of their cosh curve in $y = x$, labelled.
	Symmetry	**B1**	Equation $y = x$ stated; the line need not be drawn.
		[4]	
17(ii)	$[\sinh x]_0^k = \sinh k - \sinh 0$	**M1**	Integrates to $\pm \sinh x$ and substitutes both limits.
	$= \sinh k$	**A1**	
		[2]	
17(iii)	Reflects in $y = x$: the two regions together fill the rectangle $0 \leq x \leq \cosh k$, $0 \leq y \leq k$	**M1**	Relates the two areas, by reflection in $y = x$, to the rectangle of sides k and $\cosh k$; a labelled diagram suffices.
	$\int_0^k \cosh x \, \mathrm{d}x$ $+ \int_1^{\cosh k} \operatorname{arcosh} x \, \mathrm{d}x = k \cosh k$	**A1**	Correct relation oe, such as the arcosh integral as $k \cosh k$ minus the cosh integral.
	"$\sinh k$" $+ \int_1^{\cosh k} \operatorname{arcosh} x \, \mathrm{d}x = k \cosh k$	**M1**	Substitutes their result from (ii), of the form $\pm \sinh k$, into their area relation.

Question 17 continued

Question	Scheme	Marks	Notes
	$\int_1^{\cosh k} \operatorname{arcosh} x \, \mathrm{d}x = k\cosh k - \sinh k$	**A1 AG**	Printed result from a correct area relation and $\sinh k$, without error. Working backwards from the result earns nothing.
		[4]	
	Total for Question 17	**10**	

Question	Scheme	Marks	Notes
18(a)	$\frac{2y}{b^2}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2x}{a^2} = 0$	**M1**	Differentiates implicitly, $y^2 \to ky\frac{\mathrm{d}y}{\mathrm{d}x}$, or parametrically as $\frac{\mathrm{d}y}{\mathrm{d}\theta} \div \frac{\mathrm{d}x}{\mathrm{d}\theta}$. Condone coefficient and sign slips.
	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b^2x}{a^2y}$	**A1**	oe, including $\frac{b\sinh\theta}{a\cosh\theta}$ parametrically.
	$y - b\cosh\theta = \frac{b\sinh\theta}{a\cosh\theta}(x - a\sinh\theta)$	**M1**	Substitutes the point into their $\frac{\mathrm{d}y}{\mathrm{d}x}$ and forms a straight line through $(a\sinh\theta, b\cosh\theta)$ with that gradient.
	$ay\cosh\theta - bx\sinh\theta$ $= ab(\cosh^2\theta - \sinh^2\theta) = ab$	**A1 AG**	Correct substituted tangent equation leading without error, using $\cosh^2\theta - \sinh^2\theta = 1$, to the printed form.
		[4]	
18(b)	$a\left(\frac{b}{a}x\right)\cosh\theta - bx\sinh\theta = ab$ $\Rightarrow x = \ldots$	**M1**	Substitutes $y = \frac{b}{a}x$ (or $x = \frac{a}{b}y$) into the tangent and solves for a coordinate.
	$P = (ae^{\theta},\ be^{\theta})$	**A1**	oe, such as $\left(\frac{a}{\cosh\theta - \sinh\theta}, \frac{b}{\cosh\theta - \sinh\theta}\right)$.
		[2]	
18(c)	$y = b \Rightarrow ab\cosh\theta - bx\sinh\theta = ab$ $\Rightarrow x = \ldots$	**M1**	Uses $y = b$ for ℓ_2, substitutes into ℓ_1 and solves for x. No justification of $y = b$ needed.
	$Q = \left(\frac{a(\cosh\theta - 1)}{\sinh\theta},\ b\right)$	**A1**	oe, such as $x = a\tanh\frac{\theta}{2}$.
		[2]	
18(d)	$x = \frac{1}{2}\left(ae^{\theta} + \frac{a(\cosh\theta - 1)}{\sinh\theta}\right)$ $y = \frac{1}{2}(be^{\theta} + b)$	**M1 A1**	**M1** Finds the midpoint of their P and Q, averaging both coordinates. **A1** Both coordinates correct, oe.
	$t = e^{\theta}$: $\cosh\theta = \frac{t + t^{-1}}{2}$, $\sinh\theta = \frac{t - t^{-1}}{2}$	**M1**	Uses exponential definitions to write their midpoint coordinates in terms of e^{θ} alone.
	$\frac{\cosh\theta - 1}{\sinh\theta} = \frac{t^2 - 2t + 1}{t^2 - 1} = \frac{t - 1}{t + 1}$	**A1**	oe; implied by a correct x in t alone.
	$x = \frac{a(t^2 + 2t - 1)}{2(t + 1)}$, $y = \frac{b}{2}(t + 1)$ $2ay^2 - 2bxy = 2a\left(\frac{b}{2}(t + 1)\right)^2$ $- 2b \cdot \frac{a(t^2 + 2t - 1)}{2(t + 1)} \cdot \frac{b}{2}(t + 1)$	**dM1**	Eliminates t: substitutes their x and y in t into $2ay^2 - 2bxy$, or makes t the subject of y and substitutes into x.

Question 18 continued

Question	Scheme	Marks	Notes
	$= \frac{ab^2}{2}\left((t+1)^2 - (t^2+2t-1)\right)$ $= ab^2$	**A1 AG**	Correct substituted expression or equation leading without error to ab^2, or the printed equation. No comment on the branch needed.
		[6]	
	Total for Question 18	**14**	