## Questions

**1.** **(a)** Prove by induction that, for all integers $n \geq 1$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

**[4]**

**(b)** Hence, or otherwise, write down a factorised expression for the sum of the first $2n$ cubes

$$1^3 + 2^3 + 3^3 + \cdots + (2n)^3$$

**[1]**

**(c)** Use the formula in part (a) to write down a factorised expression for the sum of the first $n$ even cubes

$$2^3 + 4^3 + 6^3 + \cdots + (2n)^3$$

**[2]**

**(d)** Hence, or otherwise, show that

$$1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = an^2(bn^2 - 1)$$

where $a$ and $b$ are rational numbers to be determined. **[3]**

**2.** Prove by induction that, for $n \geq 1$

$$\sum_{r=1}^{n} r\,2^r = 2 + (n-1)2^{n+1}$$

**[5]**

**3.** Prove by induction that for all positive integers $n$

$$\sum_{r=1}^{n} \log\left(1 - \frac{1}{4r^2}\right) = \log\left(\frac{(2n)!(2n+1)!}{2^{4n}(n!)^4}\right)$$ **[6]**

**4.** Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$ **[5]**

**5.** Prove by induction that, for $n \geq 1$

$$\sum_{r=1}^{n} (2^r + 2 \times 3^r) = 2^{n+1} + 3^{n+1} - 5$$

**[5]**

**6.** Prove by induction that for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n} r3^{r-1} = \frac{1 + (2n-1)3^n}{4}$$

**[6]**

**7.** **(a)** Prove by induction that, for all positive integers $n$

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$$ **[6]**

**(b)** Hence, show that, for all positive integers $n$

$$\sum_{r=n+1}^{2n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{(an+b)(cn+d)}$$

where $a$, $b$, $c$ and $d$ are integers to be determined. **[3]**

**8.** Prove by induction that, for $n \geq 1$

$$\sum_{r=1}^{n} \frac{r}{2^r} = 2 - \frac{n+2}{2^n}$$

**[5]**

**9.** **(a)** Prove by induction that for all positive integers $n$

$$\sum_{r=1}^{n}(2r-1)^3 = n^2(2n^2-1)$$

**[6]**

**(b)** Given that

$$\sum_{r=1}^{2n}(2r-1)^3 = k\sum_{r=1}^{n}(2r-1)^3$$

show that

$$n^2 = \frac{k-4}{2(k-16)}$$

**[5]**

**10.** Prove by induction that for all positive integers $n$,

$$\sum_{r=1}^{n} r(r+1)(r+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$$

**[6]**