Questions

1. Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}^n = \begin{pmatrix} 2^n & 3^n - 2^n \\ 0 & 3^n \end{pmatrix}$$

[5]

2. The matrix A is given by

$$A = \begin{pmatrix} 4 & 2 \\ -2 & 0 \end{pmatrix}$$

(a) Prove by induction that

$$A^n = 2^n \begin{pmatrix} n+1 & n \\ -n & 1-n \end{pmatrix}$$

for all positive integers n. **[6]**

(b) The matrix B is given by

$$B = (A^n)^{-1}$$

Hence find B in terms of n. **[4]**

3. The matrix A is given by

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

Prove by induction that, for $n \geq 1$,

$$A^n = \begin{pmatrix} 2^n & 0 & 0 \\ n\,2^{n-1} & 2^n & 0 \\ \frac{n(n-1)}{2}\,2^{n-2} & n\,2^{n-1} & 2^n \end{pmatrix}$$

[7]

4. The matrix $M = \begin{pmatrix} 3c & 1 \\ 0 & 3c \end{pmatrix}$, where c is a positive constant.

(a) Prove by induction that for all positive integers n,

$$M^n = \begin{pmatrix} (3c)^n & n(3c)^{n-1} \\ 0 & (3c)^n \end{pmatrix}$$

[7]

(b) Given that $\det(M^n) = 144^n$, find the value of c. **[5]**

5. For $n \in \mathbb{Z}^+$, show, using mathematical induction, that

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}^n = \begin{pmatrix} (-1)^n & -n(-1)^n & \frac{n(n-1)}{2}(-1)^n \\ 0 & (-1)^n & -n(-1)^n \\ 0 & 0 & (-1)^n \end{pmatrix}$$

[5]

6. The Fibonacci sequence (F_n) is defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$.
You are given that

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

Prove by induction that

$$M^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

for all integers $n \geq 1$. **[5]**

7.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Prove by induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 1 & 2^n - 1 & 1 + (n-2)2^{n-1} \\ 0 & 2^n & n2^{n-1} \\ 0 & 0 & 2^n \end{pmatrix}$$

[5]

8. Prove by mathematical induction that, for $n \in \mathbb{N}$,

$$\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}^n = 2^{n-1} \begin{pmatrix} n+2 & -n \\ n & 2-n \end{pmatrix}$$

[6]