Questions

1. It is given that

$$y = x^2 e^{ax}$$

where a is a constant.

Prove by mathematical induction that, for all integers $n \geq 2$,

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = a^{n-2}\left(a^2x^2 + 2nax + n(n-1)\right)e^{ax}$$ **[6]**

2. Prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^3e^{-x}) = (-1)^n e^{-x}\big(x^3 - 3nx^2 + 3n(n-1)x - n(n-1)(n-2)\big)$$ **[7]**

3. Given that

$$y = e^{-2x} \cosh 3x$$

prove by induction that for $n \in \mathbb{N}$

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = e^{-2x} \left(\frac{1 + (-5)^n}{2} \cosh 3x + \frac{1 - (-5)^n}{2} \sinh 3x \right)$$ **[6]**

4. Let

$$y = x^2 \cosh x$$

Prove by induction that, for all integers $n \geq 1$,

$$\frac{\mathrm{d}^{2n}y}{\mathrm{d}x^{2n}} = x^2 \cosh x + 4nx \sinh x + 2n(2n-1)\cosh x$$

[6]

5. A function is defined by $y = f(t)$ where $f(t) = (1 + at)\ln(1 + at)$ and a is a constant.

(a) By considering $\frac{\mathrm{d}^2y}{\mathrm{d}t^2}$, $\frac{\mathrm{d}^3y}{\mathrm{d}t^3}$, $\frac{\mathrm{d}^4y}{\mathrm{d}t^4}$ and $\frac{\mathrm{d}^5y}{\mathrm{d}t^5}$, make a conjecture for a general formula for $\frac{\mathrm{d}^ny}{\mathrm{d}t^n}$ in terms of n, a and t for any integer $n \geq 2$. **[3]**

(b) Use induction to prove the formula conjectured in part (a). **[4]**

(c) In the case when $f(t) = (1 + 3t)\ln(1 + 3t)$, find the rate at which the 6^{th} derivative of $f(t)$ is varying when $t = \frac{5}{3}$. **[2]**

6. The function $f(x)$ is defined by $f(x) = (1+x)^{-\frac{1}{2}}$, for $x > -1$.

(a) Prove by mathematical induction that the nth derivative of $f(x)$, $f^{(n)}(x)$, for all $n \geq 1$, is given by

$$f^{(n)}(x) = \frac{(-1)^n(2n)!}{2^{2n}n!(1+x)^{n+\frac{1}{2}}}$$

[4]

(b) Hence prove that

$$(1+x)^{-\frac{1}{2}} = 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \ldots + (-1)^n\frac{(2n)!}{2^{2n}(n!)^2}x^n + \ldots$$

for $-1 < x \leq 1$. **[3]**

7. It is given that

$$y = (ax+1)^2 \ln(ax+1)$$

where a is a positive constant. Prove by mathematical induction that, for every integer $n \geq 3$,

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = 2(-1)^{n-3} \frac{(n-3)!a^n}{(ax+1)^{n-2}}$$

[6]

8. The function f is such that $f''(x) = -f(x)$.

Prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}\left(x^2 f(x)\right) = (-1)^n \left(x^2 f(x) - 4nxf'(x) - 2n(2n-1)f(x)\right)$$ **[7]**