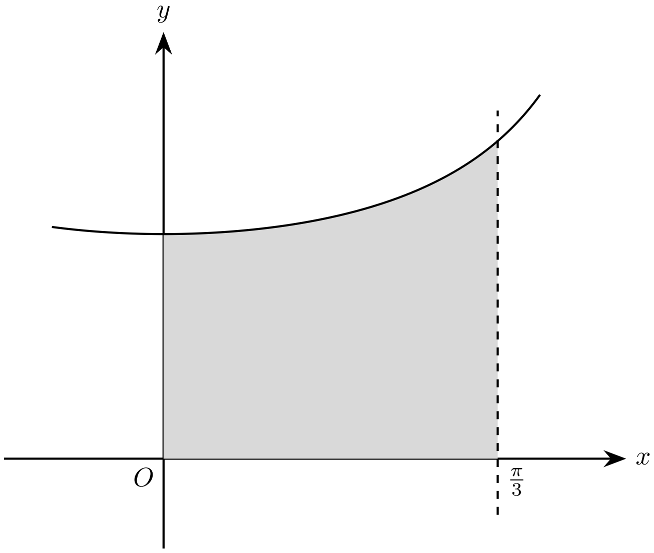
Questions

1. (a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(\sec x + \tan x) = \sec x$$ **[2]**



The shaded region is enclosed by the curve $y = \sqrt{\sec x}$, the coordinate axes and the line $x = \frac{\pi}{3}$.

The region is rotated completely about the x-axis.

(b) Find the exact volume of the solid generated. **[3]**

Solution

(a) Let $u = \sec x + \tan x$. Differentiating,

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \sec x \tan x + \sec^2 x = \sec x(\tan x + \sec x) = u \sec x$$

Hence, by the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(\sec x + \tan x) = \frac{1}{u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{u}\cdot u\sec x = \sec x$$

as required.

(b) Since the region is rotated about the x-axis, the volume is

$$V = \pi\int_a^b y^2\,\mathrm{d}x$$

Here the limits are $x = 0$ to $x = \frac{\pi}{3}$, and

$$y = \sqrt{\sec x} \implies y^2 = \sec x$$

So

$$V = \pi\int_0^{\pi/3} \sec x\,\mathrm{d}x$$

From part **(a)**,

$$\int \sec x\,\mathrm{d}x = \ln|\sec x + \tan x| + C$$

so

$$V = \pi \left[\ln(\sec x + \tan x)\right]_0^{\pi/3}$$

Now substitute the limits:

$$V = \pi \left(\ln\left(\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right) - \ln(\sec 0 + \tan 0)\right)$$

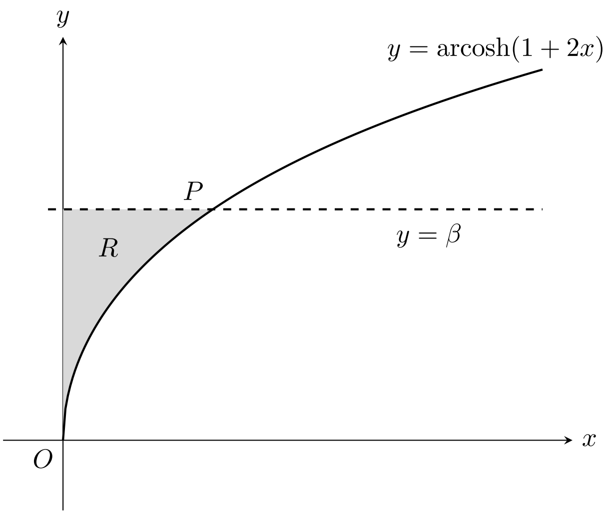
$$\sec\frac{\pi}{3} = 2, \qquad \tan\frac{\pi}{3} = \sqrt{3}, \qquad \sec 0 = 1, \qquad \tan 0 = 0$$

Hence

$$V = \pi \left(\ln(2+\sqrt{3}) - \ln 1\right)$$

$$V = \pi \ln(2+\sqrt{3})$$

So the exact volume is $\pi \ln(2+\sqrt{3})$.



2. The diagram above shows a sketch of part of the curve with equation

$$y = \operatorname{arcosh}(1 + 2x) \qquad x \geq 0$$

and the straight line with equation $y = \beta$.

The line and the curve intersect at the point with coordinates (α, β).

Given that $\beta = \ln(2 + \sqrt{3})$,

(a) show that $\alpha = \dfrac{1}{2}$. **[3]**

The finite region R, shown shaded in the diagram above, is bounded by the curve with equation $y = \operatorname{arcosh}(1 + 2x)$, the y-axis and the line with equation $y = \beta$.

The region R is rotated through 2π radians about the y-axis.

(b) Use calculus to find the exact value of the volume of the solid generated. **[6]**

Solution

(a) At the point of intersection, (α, β) lies on the curve

$$y = \operatorname{arcosh}(1 + 2x)$$

so

$$\beta = \operatorname{arcosh}(1 + 2\alpha)$$

Applying cosh to both sides gives

$$\cosh \beta = 1 + 2\alpha$$

Given

$$\beta = \ln(2 + \sqrt{3})$$

and using

$$\operatorname{arcosh} 2 = \ln(2 + \sqrt{2^2 - 1}) = \ln(2 + \sqrt{3})$$

we have

$$\beta = \operatorname{arcosh} 2$$

so

$$\cosh \beta = 2$$

Therefore

$$1 + 2\alpha = 2$$

and hence

$$2\alpha = 1$$

so

$$\alpha = \frac{1}{2}$$

So the required value is $\alpha = \frac{1}{2}$.

(b) First write x in terms of y.

From

$$y = \operatorname{arcosh}(1 + 2x)$$

we get

$$\cosh y = 1 + 2x$$

so

$$x = \frac{\cosh y - 1}{2}$$

When $x = 0$,

$$y = \operatorname{arcosh}(1) = 0$$

and the top of the region is $y = \beta$. Hence $0 \leq y \leq \beta$.

When the region is rotated about the y-axis, the volume is

$$V = \pi \int_0^\beta x^2 \, \mathrm{d}y$$

So

$$V = \pi \int_0^\beta \left(\frac{\cosh y - 1}{2} \right)^2 \mathrm{d}y = \frac{\pi}{4} \int_0^\beta (\cosh y - 1)^2 \, \mathrm{d}y$$

Now expand:

$$(\cosh y - 1)^2 = \cosh^2 y - 2\cosh y + 1$$

and use

$$\cosh^2 y = \frac{\cosh 2y + 1}{2}$$

to get

$$\begin{aligned} (\cosh y - 1)^2 &= \frac{\cosh 2y + 1}{2} - 2\cosh y + 1 \\ &= \frac{1}{2}\cosh 2y - 2\cosh y + \frac{3}{2} \end{aligned}$$

Therefore

$$\begin{aligned} V &= \frac{\pi}{4} \int_0^\beta \left(\frac{1}{2}\cosh 2y - 2\cosh y + \frac{3}{2} \right) \mathrm{d}y \\ &= \frac{\pi}{4} \left[\frac{1}{4}\sinh 2y - 2\sinh y + \frac{3}{2}y \right]_0^\beta \end{aligned}$$

Now use the result from part (a), or equivalently $\beta = \operatorname{arcosh} 2$, so

$$\cosh \beta = 2$$

Hence

$$\sinh \beta = \sqrt{\cosh^2 \beta - 1} = \sqrt{4 - 1} = \sqrt{3}$$

since $\beta > 0$.

Also,

$$\sinh 2\beta = 2\sinh\beta\cosh\beta = 2(\sqrt{3})(2) = 4\sqrt{3}$$

Substituting:

$$\begin{aligned} V &= \frac{\pi}{4}\left[\frac{1}{4}(4\sqrt{3}) - 2(\sqrt{3}) + \frac{3}{2}\beta\right] \\ &= \frac{\pi}{4}\left(\sqrt{3} - 2\sqrt{3} + \frac{3}{2}\beta\right) \\ &= \frac{\pi}{4}\left(-\sqrt{3} + \frac{3}{2}\beta\right) \\ &= \frac{\pi}{8}(3\beta - 2\sqrt{3}) \end{aligned}$$

Finally, since $\beta = \ln(2+\sqrt{3})$,

$$V = \frac{\pi}{8}\left(3\ln(2+\sqrt{3}) - 2\sqrt{3}\right)$$

So the exact volume is

$$\frac{\pi}{8}\left(3\ln(2+\sqrt{3}) - 2\sqrt{3}\right)$$

3. The equation of a curve is

$$y = \sqrt{\frac{k-x}{k+x}}$$

where k is a positive constant. The region enclosed by the curve, the x-axis, the y-axis and the line $x = k$ is rotated through 2π radians about the x-axis.

Given that the volume of the solid of revolution formed is 1 unit^3, find the exact value of k. **[4]**

Solution

When the region is rotated about the x-axis, the volume is given by

$$V = \pi \int_0^k y^2 \, \mathrm{d}x$$

Here

$$y = \sqrt{\frac{k-x}{k+x}}$$

so

$$y^2 = \frac{k-x}{k+x}$$

Given that the volume is 1 unit^3,

$$1 = \pi \int_0^k \frac{k-x}{k+x} \, \mathrm{d}x$$

First simplify the integrand:

$$\frac{k-x}{k+x} = \frac{-(k+x)+2k}{k+x} = -1 + \frac{2k}{k+x}$$

So

$$\begin{aligned} 1 &= \pi \int_0^k \left(-1 + \frac{2k}{k+x}\right) \mathrm{d}x \\ &= \pi \left[-x + 2k\ln(k+x)\right]_0^k \\ &= \pi \left([-k + 2k\ln(2k)] - [0 + 2k\ln k]\right) \\ &= \pi \left(-k + 2k\big(\ln(2k) - \ln k\big)\right) \\ &= \pi \left(-k + 2k\ln 2\right) \\ &= \pi k(2\ln 2 - 1) \end{aligned}$$

Hence

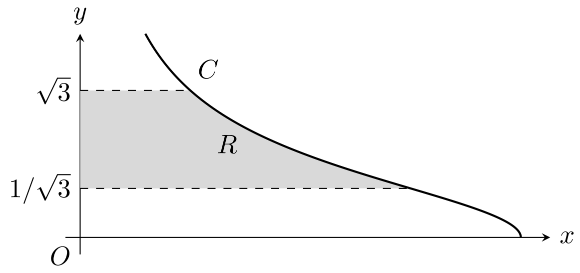
$$k = \frac{1}{\pi(2\ln 2 - 1)}$$

Therefore, the exact value of k is

$$\frac{1}{\pi(2\ln 2 - 1)}$$

4. **(a)** Find

$$\int \sin^2\theta \, d\theta$$ **[2]**



The diagram shows part of the curve C with parametric equations $x = 6\sin^2\theta$, $y = \cot\theta$, $0 < \theta \leq \frac{\pi}{2}$.

The finite region R shown in the diagram is bounded by C, the lines $y = \dfrac{1}{\sqrt{3}}$, $y = \sqrt{3}$ and the y-axis. Region R is rotated through 2π radians about the y-axis to form a solid of revolution.

(b) Show that the volume of the solid of revolution formed is given by the integral

$$k\int_a^b \sin^2\theta \, d\theta$$

where a, b and k are constants to be found. **[5]**

(c) Hence find the exact value of this volume, giving your answer in the form $p\pi^2$, where p is a constant to be found. **[3]**

Solution

(a) Use the identity

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Then

$$\begin{aligned}\int \sin^2\theta \, d\theta &= \int \frac{1-\cos 2\theta}{2} \, d\theta \\ &= \frac{1}{2}\int 1 \, d\theta - \frac{1}{2}\int \cos 2\theta \, d\theta \\ &= \frac{\theta}{2} - \frac{1}{2}\cdot\frac{\sin 2\theta}{2} + C \\ &= \frac{\theta}{2} - \frac{\sin 2\theta}{4} + C\end{aligned}$$

Hence

$$\int \sin^2\theta \, d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4} + C$$

(b) When the region is rotated about the y-axis, the volume is

$$V = \pi\int x^2 \, dy$$

Here

$$x = 6\sin^2\theta, \qquad y = \cot\theta$$

so

$$\frac{dy}{d\theta} = -\operatorname{cosec}^2\theta$$

We now find the θ-limits.

When $y = \sqrt{3}$,

$$\cot\theta = \sqrt{3} \implies \tan\theta = \frac{1}{\sqrt{3}}$$

so $\theta = \dfrac{\pi}{6}$.

When $y = \dfrac{1}{\sqrt{3}}$,

$$\cot\theta = \frac{1}{\sqrt{3}} \implies \tan\theta = \sqrt{3}$$

so $\theta = \dfrac{\pi}{3}$.

Therefore

$$\begin{aligned} V &= \pi\int_{1/\sqrt{3}}^{\sqrt{3}} x^2\,\mathrm{d}y \\ &= \pi\int_{\pi/3}^{\pi/6} (6\sin^2\theta)^2 \frac{\mathrm{d}y}{\mathrm{d}\theta}\,\mathrm{d}\theta \\ &= \pi\int_{\pi/3}^{\pi/6} (6\sin^2\theta)^2(-\operatorname{cosec}^2\theta)\,\mathrm{d}\theta \\ &= 36\pi\int_{\pi/3}^{\pi/6} -\sin^2\theta\,\mathrm{d}\theta \\ &= 36\pi\int_{\pi/6}^{\pi/3} \sin^2\theta\,\mathrm{d}\theta \end{aligned}$$

So the required form is

$$k\int_a^b \sin^2\theta\,\mathrm{d}\theta$$

with

$$k = 36\pi, \qquad a = \frac{\pi}{6}, \qquad b = \frac{\pi}{3}$$

(c) Using parts (a) and (b),

$$\begin{aligned} V &= 36\pi\int_{\pi/6}^{\pi/3} \sin^2\theta\,\mathrm{d}\theta \\ &= 36\pi\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{\pi/6}^{\pi/3} \\ &= 36\pi\left(\frac{\pi}{6} - \frac{\sin(2\pi/3)}{4} - \frac{\pi}{12} + \frac{\sin(\pi/3)}{4}\right) \end{aligned}$$

Since

$$\sin\frac{2\pi}{3} = \sin\frac{\pi}{3}$$

the sine terms cancel, giving

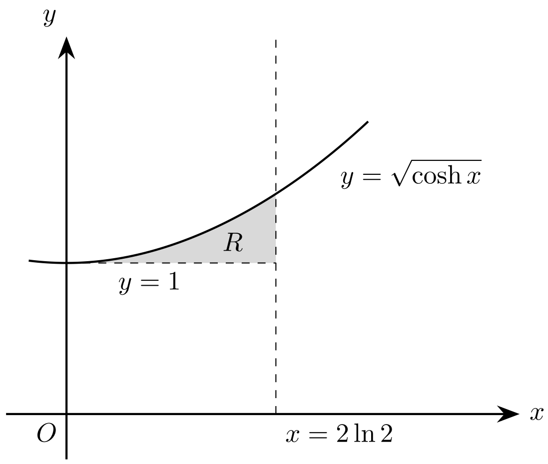
$$\begin{aligned} V &= 36\pi\left(\frac{\pi}{6} - \frac{\pi}{12}\right) \\ &= 36\pi\cdot\frac{\pi}{12} \\ &= 3\pi^2 \end{aligned}$$

Hence the exact volume is

$$3\pi^2$$

so $p = 3$.

5. **(a)** Show that $\sinh(2\ln 2) = \dfrac{15}{8}$ **[2]**



The region R is bounded by the curve with equation $y = \sqrt{\cosh x}$, the line $y = 1$ and the line $x = 2\ln 2$, as shown in the diagram. The units of the axes are centimetres.

A designer models a hollow glass ornament as the solid formed by rotating R completely about the x-axis.

(b) Determine, according to the model, the exact volume of the ornament. **[4]**

Solution

(a) Using

$$\sinh u = \frac{e^u - e^{-u}}{2}$$

with $u = 2\ln 2$,

$$\begin{aligned}\sinh(2\ln 2) &= \frac{e^{2\ln 2} - e^{-2\ln 2}}{2} \\ &= \frac{e^{\ln 4} - e^{-\ln 4}}{2} \\ &= \frac{4 - \frac{1}{4}}{2} \\ &= \frac{\frac{16}{4} - \frac{1}{4}}{2} = \frac{15/4}{2} = \frac{15}{8}\end{aligned}$$

Hence,

$$\sinh(2\ln 2) = \frac{15}{8}$$

(b) First find the left-hand limit of the region by solving where the curve meets $y = 1$:

$$\begin{aligned}\sqrt{\cosh x} &= 1 \\ \cosh x &= 1\end{aligned}$$

and this gives $x = 0$.

So the region extends from $x = 0$ to $x = 2\ln 2$.

When the region is rotated about the x-axis, the volume is

$$V = \pi \int_a^b \left((\text{upper curve})^2 - (\text{lower curve})^2\right)\,\mathrm{d}x$$

Here, the upper curve is $y = \sqrt{\cosh x}$ and the lower curve is $y = 1$, so

$$\begin{aligned} V &= \pi \int_0^{2\ln 2} \left(\left(\sqrt{\cosh x} \right)^2 - 1^2 \right) \, \mathrm{d}x \\ &= \pi \int_0^{2\ln 2} (\cosh x - 1) \, \mathrm{d}x \end{aligned}$$

Integrating,

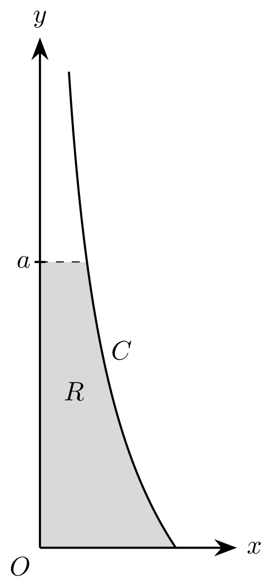
$$\begin{aligned} V &= \pi \left[\sinh x - x \right]_0^{2\ln 2} \\ &= \pi \left(\sinh(2\ln 2) - 2\ln 2 - (\sinh 0 - 0) \right) \\ &= \pi \left(\sinh(2\ln 2) - 2\ln 2 \right) \end{aligned}$$

From part (a), $\sinh(2\ln 2) = \dfrac{15}{8}$, so

$$V = \pi \left(\frac{15}{8} - 2\ln 2 \right) \text{ cm}^3$$

Therefore, the exact volume of the ornament is

$$\pi \left(\frac{15}{8} - 2\ln 2 \right) \text{ cm}^3$$



6. The curve C is given parametrically by

$$x = \frac{9}{t^4}, \qquad y = t^2 - 3, \qquad t \geq \sqrt{3}$$

The shaded region R, shown in the diagram, is enclosed by C, the x-axis, the y-axis and the line $y = a$. When R is rotated through 2π radians about the y-axis, the resulting solid has volume

$$\frac{7\pi}{8}$$

Find the exact value of a. **[8]**

Solution

First write the curve in the form x as a function of y.
From

$$y = t^2 - 3$$

we have

$$t^2 = y + 3$$

So

$$x = \frac{9}{t^4} = \frac{9}{(t^2)^2} = \frac{9}{(y+3)^2}$$

Since $t \geq \sqrt{3}$, we have $y \geq 0$, so the region runs from $y = 0$ up to $y = a$.
When the region is rotated about the y-axis, the volume is

$$V = \pi \int_0^a x^2 \, \mathrm{d}y$$

Hence

$$\begin{aligned}
V &= \pi \int_0^a \left(\frac{9}{(y+3)^2} \right)^2 \mathrm{d}y \\
&= \pi \int_0^a \frac{81}{(y+3)^4} \mathrm{d}y \\
&= 81\pi \int_0^a (y+3)^{-4} \mathrm{d}y \\
&= 81\pi \left[\frac{(y+3)^{-3}}{-3} \right]_0^a \\
&= 81\pi \left[-\frac{1}{3(y+3)^3} \right]_0^a \\
&= 81\pi \left(-\frac{1}{3(a+3)^3} + \frac{1}{3 \cdot 3^3} \right) \\
&= 81\pi \left(-\frac{1}{3(a+3)^3} + \frac{1}{81} \right) \\
&= -\frac{27\pi}{(a+3)^3} + \pi \\
&= \pi \left(1 - \frac{27}{(a+3)^3} \right)
\end{aligned}$$

We are told that the volume is $\frac{7\pi}{8}$, so

$$\begin{aligned}
\pi \left(1 - \frac{27}{(a+3)^3} \right) &= \frac{7\pi}{8} \\
1 - \frac{27}{(a+3)^3} &= \frac{7}{8} \\
\frac{27}{(a+3)^3} &= \frac{1}{8} \\
(a+3)^3 &= 216
\end{aligned}$$

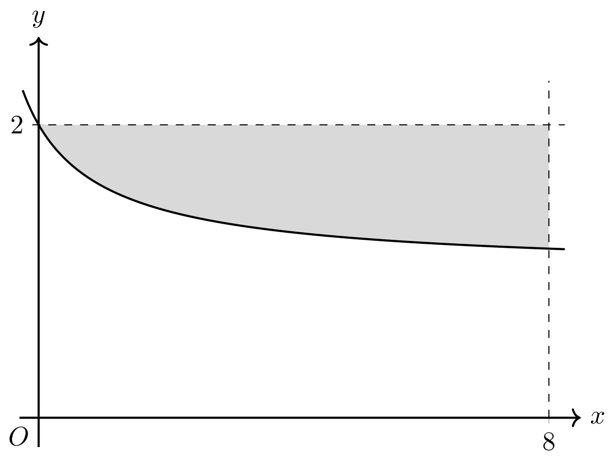
Now $a \geq 0$, so $a + 3 > 0$, and therefore

$$a + 3 = 6$$

Thus

$$a = 3$$

The exact value of a is 3.



7. The diagram shows the region bounded by the curve $y = \sqrt{\dfrac{x+4}{x+1}}$, the line $y = 2$ and the line $x = 8$. The curve meets the line $y = 2$ at the point $(0, 2)$.

Find the exact volume of the solid formed when this region is rotated through 360° about the x-axis. **[6]**

Solution

The region is between the line $y = 2$ and the curve

$$y = \sqrt{\frac{x+4}{x+1}}$$

from $x = 0$ to $x = 8$.
When this region is rotated about the x-axis, the volume is

$$V = \pi \int_0^8 \left(2^2 - \left(\sqrt{\frac{x+4}{x+1}} \right)^2 \right) \mathrm{d}x$$

So

$$V = \pi \int_0^8 \left(4 - \frac{x+4}{x+1} \right) \mathrm{d}x$$

Simplify the integrand:

$$\begin{aligned} 4 - \frac{x+4}{x+1} &= \frac{4(x+1) - (x+4)}{x+1} \\ &= \frac{4x + 4 - x - 4}{x+1} \\ &= \frac{3x}{x+1} \end{aligned}$$

Hence

$$V = \pi \int_0^8 \frac{3x}{x+1} \, \mathrm{d}x = 3\pi \int_0^8 \frac{x}{x+1} \, \mathrm{d}x$$

Now write

$$\frac{x}{x+1} = 1 - \frac{1}{x+1}$$

Therefore

$$V = 3\pi \int_0^8 \left(1 - \frac{1}{x+1} \right) \mathrm{d}x$$

Integrating gives

$$V = 3\pi \left[x - \ln(x+1) \right]_0^8$$

Substitute the limits:

$$\begin{aligned} V &= 3\pi \left((8 - \ln 9) - (0 - \ln 1) \right) \\ &= 3\pi(8 - \ln 9) \end{aligned}$$

Since $\ln 9 = 2\ln 3$,

$$\begin{aligned} V &= 24\pi - 3\pi\ln 9 \\ &= 24\pi - 6\pi\ln 3 \\ &= 6\pi(4 - \ln 3) \end{aligned}$$

So the exact volume is $6\pi(4 - \ln 3)$.

8. The region in the first quadrant bounded by the curve $y = \sinh x$, the y-axis, and the line $y = 2$ is rotated through $360°$ about the x-axis.

Find the exact volume of revolution generated, expressing your answer in a form involving a logarithm. **[7]**

Solution

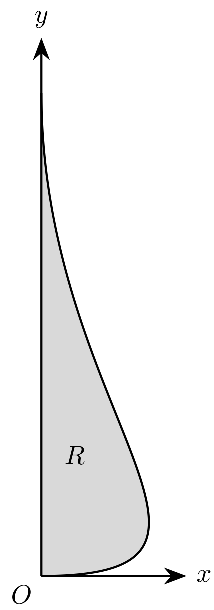
Let the curve $y = \sinh x$ meet the line $y = 2$ at $x = a$.
Then
$$\sinh a = 2$$
To find a in logarithmic form,
$$\begin{aligned} \sinh a = 2 &\implies \frac{e^a - e^{-a}}{2} = 2 \\ &\implies e^a - e^{-a} = 4 \\ &\implies e^{2a} - 1 = 4e^a \end{aligned}$$
Let $u = e^a$. Then $u > 0$ and
$$u^2 - 4u - 1 = 0$$
So
$$u = \frac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5}$$
Since $u > 0$, we take
$$u = 2 + \sqrt{5}$$
Hence
$$a = \ln(2 + \sqrt{5})$$
For $0 \le x \le a$, the region lies between $y = \sinh x$ and $y = 2$. When it is rotated about the x-axis, the volume is
$$V = \pi \int_0^a \left(2^2 - (\sinh x)^2\right) \, \mathrm{d}x$$
So
$$V = \pi \int_0^a \left(4 - \sinh^2 x\right) \, \mathrm{d}x$$
Use
$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$
Then
$$\begin{aligned} V &= \pi \int_0^a \left(4 - \frac{\cosh 2x - 1}{2}\right) \mathrm{d}x \\ &= \pi \int_0^a \left(\frac{9}{2} - \frac{1}{2}\cosh 2x\right) \mathrm{d}x \\ &= \pi \left[\frac{9}{2}x - \frac{1}{4}\sinh 2x\right]_0^a \end{aligned}$$
Now $\sinh a = 2$, and since $a > 0$,
$$\cosh a = \sqrt{1 + \sinh^2 a} = \sqrt{1+4} = \sqrt{5}$$
Therefore
$$\sinh 2a = 2 \sinh a \cosh a = 2(2)(\sqrt{5}) = 4\sqrt{5}$$
Substituting into the volume,
$$\begin{aligned} V &= \pi \left(\frac{9}{2}a - \frac{1}{4}\sinh 2a\right) \\ &= \pi \left(\frac{9}{2}a - \frac{1}{4}(4\sqrt{5})\right) \\ &= \pi \left(\frac{9}{2}a - \sqrt{5}\right) \end{aligned}$$

Finally, using $a = \ln(2+\sqrt{5})$,

$$V = \pi\left(\frac{9}{2}\ln(2+\sqrt{5}) - \sqrt{5}\right)$$

So the exact volume is

$$\pi\left(\frac{9}{2}\ln(2+\sqrt{5}) - \sqrt{5}\right)$$



9. Part of the side profile of a solid wooden ornament is shown in the diagram.
The outline is modelled by the curve with parametric equations

$$x = 12t(1-t)^2, \quad y = 8t^2, \qquad 0 \le t \le 1$$

The ornament is formed by rotating the shaded region through 2π radians about the y-axis.

Use the model to find the volume of the ornament. **[7]**

Solution

For a solid formed by rotating a region about the y-axis, we use

$$V = \pi \int x^2 \, \mathrm{d}y$$

Here

$$x = 12t(1-t)^2, \qquad y = 8t^2, \qquad 0 \le t \le 1$$

Also,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 16t$$

Since $0 \le t \le 1$, we have $\frac{\mathrm{d}y}{\mathrm{d}t} \ge 0$, so y increases through the interval and we can write

$$\begin{aligned} V &= \pi \int_0^1 x^2 \frac{\mathrm{d}y}{\mathrm{d}t} \, \mathrm{d}t \\ &= \pi \int_0^1 \left(12t(1-t)^2\right)^2 (16t) \, \mathrm{d}t \\ &= \pi \int_0^1 144t^2(1-t)^4 \cdot 16t \, \mathrm{d}t \\ &= 2304\pi \int_0^1 t^3(1-t)^4 \, \mathrm{d}t \end{aligned}$$

Now expand

$$(1-t)^4 = 1 - 4t + 6t^2 - 4t^3 + t^4$$

So

$$t^3(1-t)^4 = t^3 - 4t^4 + 6t^5 - 4t^6 + t^7$$

Hence

$$\begin{aligned} V &= 2304\pi \int_0^1 \left(t^3 - 4t^4 + 6t^5 - 4t^6 + t^7\right) \, \mathrm{d}t \\ &= 2304\pi \left[\frac{t^4}{4} - \frac{4t^5}{5} + t^6 - \frac{4t^7}{7} + \frac{t^8}{8}\right]_0^1 \\ &= 2304\pi \left(\frac{1}{4} - \frac{4}{5} + 1 - \frac{4}{7} + \frac{1}{8}\right) \end{aligned}$$

Using a common denominator of 280,

$$\frac{1}{4} - \frac{4}{5} + 1 - \frac{4}{7} + \frac{1}{8} = \frac{70 - 224 + 280 - 160 + 35}{280} = \frac{1}{280}$$

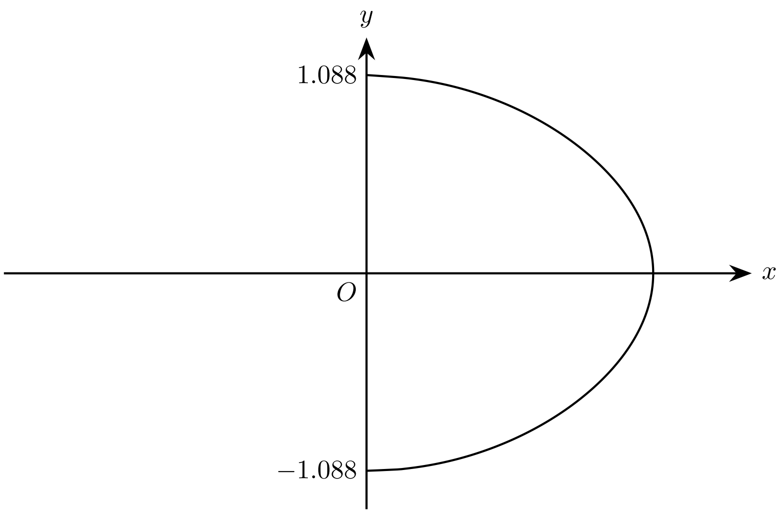
Therefore

$$\begin{aligned} V &= 2304\pi \cdot \frac{1}{280} \\ &= \frac{288\pi}{35} \end{aligned}$$

So the volume of the ornament is

$$\frac{288\pi}{35} \text{ units}^3$$

which is approximately 25.9 units^3.



10. A chocolatier makes hand-rolled chocolate drops.

The tallest drop in one tray was approximately 2.2 cm high.

The shape of this drop is modelled by rotating the curve with equation

$$20x^2 + 6y^2 + y\sin(2y) = 8, \qquad x \geq 0$$

shown in the diagram above, about the y-axis through 2π radians, where the units are cm.

Given that the y-intercepts of the curve are -1.088 and 1.088 to four significant figures,

(a) Use algebraic integration to determine, according to the model, the volume of this chocolate drop. **[6]**

The chocolatier melts down 80 chocolate drops from the same tray.

(b) Use your answer to part (a) to decide whether, in reality, there is likely to be enough chocolate to fill a mould of volume $140\,\text{cm}^3$, giving a reason. **[2]**

Solution

(a) At the y-intercepts, $x = 0$, so the limits of integration are

$$y = -1.088 \quad \text{and} \quad y = 1.088$$

From

$$20x^2 + 6y^2 + y\sin(2y) = 8$$

we rearrange to get

$$x^2 = \frac{8 - 6y^2 - y\sin(2y)}{20}$$

When this curve is rotated about the y-axis, the volume is

$$V = \pi \int_{-1.088}^{1.088} x^2 \, \mathrm{d}y$$

So

$$V = \frac{\pi}{20} \int_{-1.088}^{1.088} \left(8 - 6y^2 - y\sin(2y)\right) \, \mathrm{d}y$$

Using integration by parts for $\int -y\sin(2y)\,\mathrm{d}y$, let

$$u = y, \qquad \frac{\mathrm{d}v}{\mathrm{d}y} = -\sin(2y)$$

Then

$$\frac{\mathrm{d}u}{\mathrm{d}y} = 1, \qquad v = \frac{1}{2}\cos(2y)$$

Hence

$$\begin{aligned}\int -y\sin(2y)\,\mathrm{d}y &= uv - \int v\,\frac{\mathrm{d}u}{\mathrm{d}y}\,\mathrm{d}y \\ &= \frac{y}{2}\cos(2y) - \int \frac{1}{2}\cos(2y)\,\mathrm{d}y \\ &= \frac{y}{2}\cos(2y) - \frac{1}{4}\sin(2y)\end{aligned}$$

Therefore

$$\begin{aligned}V &= \frac{\pi}{20}\left[8y - 2y^3 + \frac{y}{2}\cos(2y) - \frac{1}{4}\sin(2y)\right]_{-1.088}^{1.088} \\ &\approx 1.764\ldots\end{aligned}$$

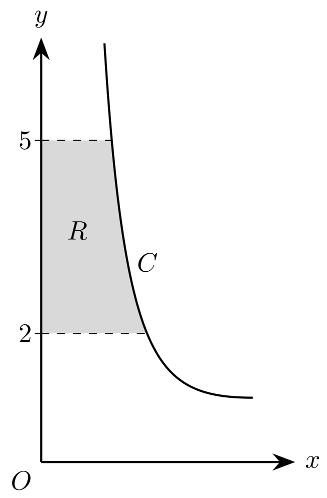
Hence, according to the model, the volume of the chocolate drop is $1.76\,\mathrm{cm}^3$.

(b) Using the answer from part (a), the volume of 80 such drops would be estimated as

$$80 \times 1.76 = 140.8\,\mathrm{cm}^3$$

This is only just above $140\,\mathrm{cm}^3$. Also, the model in part (a) is for the tallest drop in the tray, so using $1.76\,\mathrm{cm}^3$ for all 80 drops will overestimate the total amount of chocolate.

Therefore, in reality, there is unlikely to be enough chocolate to fill the $140\,\mathrm{cm}^3$ mould.



11. The diagram shows part of the curve C with parametric equations

$$x = \frac{2}{1+t}, \quad y = t^2 + 1, \qquad t \geq 0$$

The region R is bounded by the curve, the y-axis and the lines $y = 2$ and $y = 5$. Region R is rotated through 2π radians about the y-axis.

Use parametric integration to find the volume of the resulting solid of revolution. **[6]**

Solution

For the given curve,

$$x = \frac{2}{1+t}, \qquad y = t^2 + 1, \qquad t \geq 0$$

First find the values of t corresponding to the horizontal boundaries.
When $y = 2$,

$$t^2 + 1 = 2 \implies t^2 = 1 \implies t = 1$$

since $t \geq 0$.
When $y = 5$,

$$t^2 + 1 = 5 \implies t^2 = 4 \implies t = 2$$

So the required part of the curve corresponds to $1 \leq t \leq 2$.
When the region is rotated about the y-axis, the volume is

$$V = \pi \int x^2 \, \mathrm{d}y$$

Using the parameter t,

$$V = \pi \int_1^2 x^2 \frac{\mathrm{d}y}{\mathrm{d}t} \, \mathrm{d}t$$

Now

$$x^2 = \left(\frac{2}{1+t}\right)^2 = \frac{4}{(1+t)^2}$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2t$$

Hence

$$V = \pi \int_1^2 \frac{4}{(1+t)^2} \cdot 2t \, \mathrm{d}t = 8\pi \int_1^2 \frac{t}{(1+t)^2} \, \mathrm{d}t$$

Now simplify the integrand:

$$\frac{t}{(1+t)^2} = \frac{(1+t) - 1}{(1+t)^2} = \frac{1}{1+t} - \frac{1}{(1+t)^2}$$

So

$$V = 8\pi \int_1^2 \left(\frac{1}{1+t} - \frac{1}{(1+t)^2} \right) \mathrm{d}t$$

Integrating,

$$\int \frac{1}{1+t} \, \mathrm{d}t = \ln(1+t)$$

and

$$\int -\frac{1}{(1+t)^2} \, \mathrm{d}t = \frac{1}{1+t}$$

Therefore

$$V = 8\pi \left[\ln(1+t) + \frac{1}{1+t} \right]_1^2$$

Substituting the limits,

$$V = 8\pi \left(\ln 3 + \frac{1}{3} - \ln 2 - \frac{1}{2} \right)$$

So

$$V = 8\pi \left(\ln \frac{3}{2} - \frac{1}{6} \right)$$

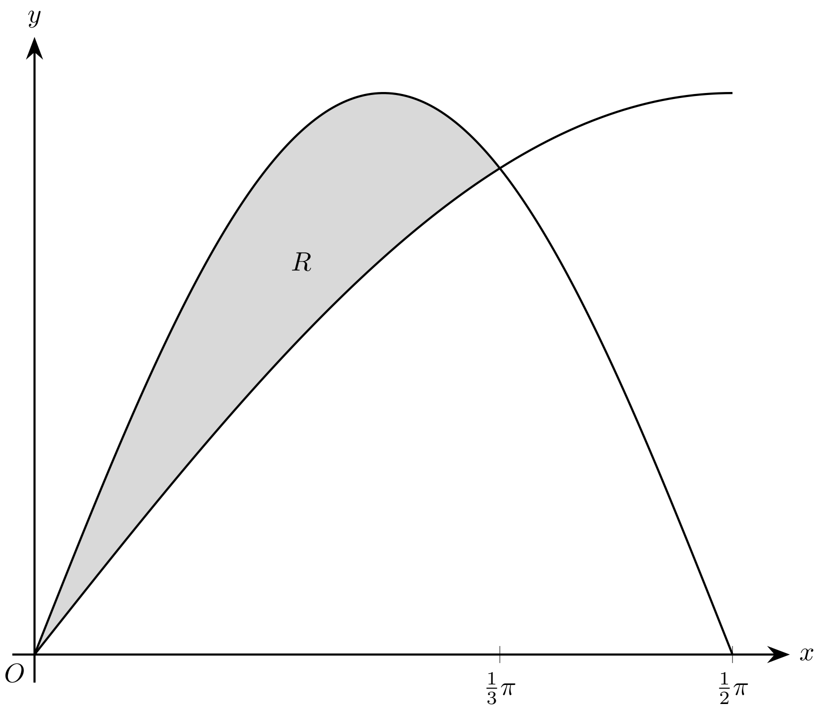
$$V = 8\pi \ln \frac{3}{2} - \frac{4\pi}{3}$$

Hence the volume of the solid is

$$\boxed{8\pi \ln \frac{3}{2} - \frac{4\pi}{3}}$$

In decimal form,

$$V \approx 6.00$$



12. The diagram shows the curves $y = \sin x$ and $y = \sin 2x$, for $0 \leq x \leq \frac{1}{2}\pi$. The shaded region R is the finite region enclosed by the curves.

Find the volume of the solid of revolution formed when R is rotated completely about the x-axis, giving your answer in terms of π. **[7]**

Solution

First find where the curves meet.

$$\sin x = \sin 2x$$

Using $\sin 2x = 2 \sin x \cos x$,

$$\sin x = 2 \sin x \cos x$$

$$\sin x(2 \cos x - 1) = 0$$

So, for $0 \leq x \leq \frac{\pi}{2}$,

$$\sin x = 0 \Rightarrow x = 0 \qquad \text{or} \qquad 2 \cos x - 1 = 0 \Rightarrow \cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}$$

Hence the shaded region is between $x = 0$ and $x = \frac{\pi}{3}$.
For $0 < x < \frac{\pi}{3}$, we have $\sin x > 0$ and $\cos x > \frac{1}{2}$, so $2 \cos x - 1 > 0$. Therefore

$$\sin x(2 \cos x - 1) > 0$$

which means

$$\sin 2x > \sin x$$

So when the region is rotated about the x-axis, the volume is

$$V = \pi \int_0^{\pi/3} \left((\sin 2x)^2 - (\sin x)^2\right) \, \mathrm{d}x$$

Now use

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Then

$$\begin{aligned}\sin^2 2x - \sin^2 x &= \frac{1-\cos 4x}{2} - \frac{1-\cos 2x}{2}\\ &= \frac{\cos 2x - \cos 4x}{2}\end{aligned}$$

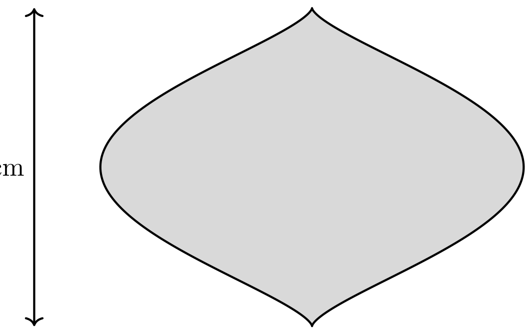
So

$$\begin{aligned}V &= \pi\int_0^{\pi/3} \frac{\cos 2x - \cos 4x}{2}\,\mathrm{d}x\\ &= \pi\left[\frac{\sin 2x}{4} - \frac{\sin 4x}{8}\right]_0^{\pi/3}\end{aligned}$$

Now substitute the limits:

$$\begin{aligned}V &= \pi\left(\frac{\sin(2\pi/3)}{4} - \frac{\sin(4\pi/3)}{8}\right)\\ &= \pi\left(\frac{\sqrt{3}/2}{4} - \frac{-\sqrt{3}/2}{8}\right)\\ &= \pi\left(\frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{16}\right)\\ &= \frac{3\sqrt{3}\pi}{16}\end{aligned}$$

The volume of the solid of revolution is $\frac{3\sqrt{3}\pi}{16}$.



13. The diagram shows the outline of a glass bead. The bead is modelled by the solid obtained when the region enclosed by a curve C is rotated about the y-axis. The actual bead has height 3 cm. The curve C has parametric equations

$$x = \sin\theta - \frac{1}{3}\sin 3\theta, \qquad y = 1 - \cos\theta, \qquad 0 \le \theta \le 2\pi$$

(a) Show that a Cartesian equation of the curve C is

$$x^2 = \frac{16}{9}y^3(2-y)^3$$

[4]

(b) Hence, using the model, find, in cm^3, the volume of the bead. **[5]**

Solution

(a) Using the identity

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

we have

$$\begin{aligned} x &= \sin\theta - \frac{1}{3}\sin 3\theta \\ &= \sin\theta - \frac{1}{3}\left(3\sin\theta - 4\sin^3\theta\right) \\ &= \sin\theta - \sin\theta + \frac{4}{3}\sin^3\theta \\ &= \frac{4}{3}\sin^3\theta \end{aligned}$$

So

$$x^2 = \frac{16}{9}\sin^6\theta$$

Also

$$y = 1 - \cos\theta$$

so

$$\cos\theta = 1 - y$$

Hence

$$\begin{aligned} \sin^2\theta &= 1 - \cos^2\theta \\ &= 1 - (1-y)^2 \\ &= 1 - (1 - 2y + y^2) \\ &= 2y - y^2 \\ &= y(2-y) \end{aligned}$$

Therefore

$$\sin^6\theta = \left(\sin^2\theta\right)^3 = (y(2-y))^3$$

and so

$$\begin{aligned}x^2 &= \frac{16}{9}\sin^6\theta \\ &= \frac{16}{9}\left(y(2-y)\right)^3 \\ &= \frac{16}{9}y^3(2-y)^3\end{aligned}$$

Thus a Cartesian equation of C is

$$x^2 = \frac{16}{9}y^3(2-y)^3$$

(b) The model has height

$$2 - 0 = 2$$

whereas the actual bead has height 3 cm, so the linear scale factor is

$$k = \frac{3}{2}$$

First find the volume of the model.
When the region is rotated about the y-axis,

$$V_{\text{model}} = \pi\int_0^2 x^2\,\mathrm{d}y$$

Using the result from part (a),

$$\begin{aligned}V_{\text{model}} &= \pi\int_0^2 \frac{16}{9}y^3(2-y)^3\,\mathrm{d}y \\ &= \frac{16\pi}{9}\int_0^2 y^3(2-y)^3\,\mathrm{d}y\end{aligned}$$

Now expand:

$$\begin{aligned}(2-y)^3 &= 8 - 12y + 6y^2 - y^3 \\ y^3(2-y)^3 &= 8y^3 - 12y^4 + 6y^5 - y^6\end{aligned}$$

So

$$\begin{aligned}V_{\text{model}} &= \frac{16\pi}{9}\int_0^2\left(8y^3 - 12y^4 + 6y^5 - y^6\right)\,\mathrm{d}y \\ &= \frac{16\pi}{9}\left[2y^4 - \frac{12}{5}y^5 + y^6 - \frac{1}{7}y^7\right]_0^2\end{aligned}$$

Substituting the limits,

$$\begin{aligned}V_{\text{model}} &= \frac{16\pi}{9}\left(2(2^4) - \frac{12}{5}(2^5) + 2^6 - \frac{1}{7}(2^7)\right) \\ &= \frac{16\pi}{9}\left(32 - \frac{384}{5} + 64 - \frac{128}{7}\right) \\ &= \frac{16\pi}{9}\cdot\frac{32}{35} \\ &= \frac{512\pi}{315}\end{aligned}$$

Volume scales as the cube of the linear scale factor, so

$$\begin{aligned}V_{\text{actual}} &= \left(\frac{3}{2}\right)^3 V_{\text{model}} \\ &= \frac{27}{8}\times\frac{512\pi}{315} \\ &= \frac{192\pi}{35}\end{aligned}$$

Therefore the volume of the bead is

$$\frac{192\pi}{35}\text{ cm}^3 \approx 17.2\text{ cm}^3$$

14. The equation of a curve is

$$y = \frac{x}{\sqrt{k^2 + x^2}}$$

where k is a positive constant.

The region in the first quadrant bounded by the curve, the x-axis, the y-axis and the line $x = k$ is rotated through 2π radians about the x-axis
Given that the volume of revolution formed is 1 unit^3, find the exact value of k **[4]**

Solution

When the region is rotated about the x-axis, the volume is

$$V = \pi \int_0^k y^2 \, \mathrm{d}x$$

Here

$$y = \frac{x}{\sqrt{k^2 + x^2}} \qquad \Rightarrow \qquad y^2 = \frac{x^2}{k^2 + x^2}$$

So

$$V = \pi \int_0^k \frac{x^2}{k^2 + x^2} \, \mathrm{d}x$$

Rewrite the integrand:

$$\frac{x^2}{k^2 + x^2} = \frac{k^2 + x^2 - k^2}{k^2 + x^2} = 1 - \frac{k^2}{k^2 + x^2}$$

Hence

$$V = \pi \int_0^k \left(1 - \frac{k^2}{k^2 + x^2}\right) \mathrm{d}x$$

Now integrate:

$$\begin{aligned}
\int \left(1 - \frac{k^2}{k^2 + x^2}\right) \mathrm{d}x &= x - k^2 \int \frac{1}{k^2 + x^2} \, \mathrm{d}x \\
&= x - k^2 \left(\frac{1}{k} \arctan \frac{x}{k}\right) \\
&= x - k \arctan \frac{x}{k}
\end{aligned}$$

Therefore

$$\begin{aligned}
V &= \pi \left[x - k \arctan \frac{x}{k}\right]_0^k \\
&= \pi \left(k - k \arctan 1\right) \\
&= \pi \left(k - k \cdot \frac{\pi}{4}\right) \\
&= \pi k \left(1 - \frac{\pi}{4}\right)
\end{aligned}$$

We are given that the volume is 1, so

$$\pi k \left(1 - \frac{\pi}{4}\right) = 1$$

Solving for k:

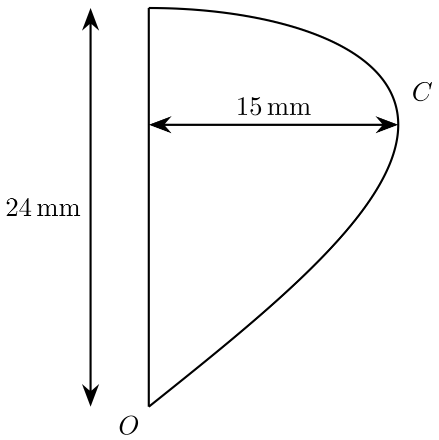
$$\begin{aligned}
k &= \frac{1}{\pi \left(1 - \frac{\pi}{4}\right)} \\
&= \frac{1}{\pi \left(\frac{4-\pi}{4}\right)} \\
&= \frac{4}{\pi(4 - \pi)}
\end{aligned}$$

Since k is positive, this is valid.
The exact value of k is

$$k = \frac{4}{\pi(4 - \pi)}$$

15. **(a)** Prove that

$$\cos 4\theta \cos\theta \equiv \frac{1}{2}(\cos 5\theta + \cos 3\theta)$$ **[3]**



The diagram shows the cross-section of a decorative ornament.

The ornament can be modelled by rotating the region enclosed by the curve C and the y-axis about the y-axis. Curve C has parametric equations

$$x = 15\sin 2\theta, \quad y = 24\sin\theta, \quad 0 \le \theta \le \frac{\pi}{2}$$

where x and y are measured in millimetres.

(b) Find the volume of the ornament. **[5]**

Solution

(a) Using the cosine addition formula,

$$\cos 5\theta = \cos(4\theta + \theta) = \cos 4\theta \cos\theta - \sin 4\theta \sin\theta$$

and

$$\cos 3\theta = \cos(4\theta - \theta) = \cos 4\theta \cos\theta + \sin 4\theta \sin\theta$$

Adding these gives

$$\begin{aligned}\cos 5\theta + \cos 3\theta &= (\cos 4\theta \cos\theta - \sin 4\theta \sin\theta) + (\cos 4\theta \cos\theta + \sin 4\theta \sin\theta)\\ &= 2\cos 4\theta \cos\theta\end{aligned}$$

Therefore

$$\cos 4\theta \cos\theta \equiv \frac{1}{2}(\cos 5\theta + \cos 3\theta)$$

So the required identity is proved.

(b) When the region is rotated about the y-axis, the volume is

$$V = \pi \int x^2 \,\mathrm{d}y$$

Using the parameter θ,

$$V = \pi \int_0^{\pi/2} x^2 \frac{\mathrm{d}y}{\mathrm{d}\theta}\,\mathrm{d}\theta$$

Now

$$x = 15\sin 2\theta, \qquad y = 24\sin\theta$$

so

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = 24\cos\theta$$

Substituting,

$$\begin{aligned} V &= \pi \int_0^{\pi/2} (15\sin 2\theta)^2 (24\cos\theta)\,\mathrm{d}\theta \\ &= 5400\pi \int_0^{\pi/2} \sin^2 2\theta \cos\theta\,\mathrm{d}\theta \end{aligned}$$

Use

$$\sin^2 2\theta = \frac{1-\cos 4\theta}{2}$$

Then

$$\begin{aligned} V &= 5400\pi \int_0^{\pi/2} \frac{1-\cos 4\theta}{2}\cos\theta\,\mathrm{d}\theta \\ &= 2700\pi \int_0^{\pi/2} (1-\cos 4\theta)\cos\theta\,\mathrm{d}\theta \\ &= 2700\pi \left(\int_0^{\pi/2} \cos\theta\,\mathrm{d}\theta - \int_0^{\pi/2} \cos 4\theta\cos\theta\,\mathrm{d}\theta \right) \end{aligned}$$

From part (a),

$$\cos 4\theta \cos\theta = \frac{1}{2}(\cos 5\theta + \cos 3\theta)$$

So

$$\begin{aligned} V &= 2700\pi \int_0^{\pi/2} \left(\cos\theta - \frac{1}{2}(\cos 5\theta + \cos 3\theta) \right)\,\mathrm{d}\theta \\ &= 2700\pi \left[\sin\theta - \frac{1}{10}\sin 5\theta - \frac{1}{6}\sin 3\theta \right]_0^{\pi/2} \end{aligned}$$

Now evaluate the bracket:

$$\sin\frac{\pi}{2} = 1, \qquad \sin\frac{5\pi}{2} = 1, \qquad \sin\frac{3\pi}{2} = -1$$

Hence

$$\begin{aligned} V &= 2700\pi \left(1 - \frac{1}{10} - \frac{1}{6}(-1)\right) \\ &= 2700\pi \left(1 - \frac{1}{10} + \frac{1}{6}\right) \\ &= 2700\pi \left(\frac{16}{15}\right) \\ &= 2880\pi \end{aligned}$$

Therefore the volume of the ornament is

$$2880\pi \text{ mm}^3$$

which is approximately

$$9.05 \times 10^3 \text{ mm}^3$$