Questions

1. (a) (i) Given that $f(x) = e^{-2x}$, find $f'(x)$ and $f''(x)$. **[2]**

(ii) Hence, find the first three terms of the Maclaurin series for e^{-2x}. **[2]**

(b) Hence, using a suitable value of x, show that

$$e^{-1/4} \approx \frac{25}{32}$$

. **[2]**

Solution

(a) (i) Differentiate $f(x) = e^{-2x}$ using the chain rule:

$$f'(x) = (-2)e^{-2x} = -2e^{-2x}$$

Differentiate again:

$$f''(x) = (-2)(-2)e^{-2x} = 4e^{-2x}$$

So

$$f'(x) = -2e^{-2x}, \qquad f''(x) = 4e^{-2x}$$

(ii) The Maclaurin series is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots$$

Now

$$f(0) = e^0 = 1, \qquad f'(0) = -2e^0 = -2, \qquad f''(0) = 4e^0 = 4$$

So

$$\begin{aligned} e^{-2x} &= 1 + x(-2) + \frac{x^2}{2}(4) + \cdots \\ &= 1 - 2x + 2x^2 + \cdots \end{aligned}$$

Hence the first three terms are

$$1 - 2x + 2x^2$$

(b) To get $e^{-1/4}$, choose x so that

$$-2x = -\frac{1}{4}$$

Hence

$$x = \frac{1}{8}$$

Using the Maclaurin expansion found in part (a),

$$e^{-2x} \approx 1 - 2x + 2x^2$$

Substituting $x = \frac{1}{8}$,

$$\begin{aligned} e^{-1/4} &= e^{-2(1/8)} \\ &\approx 1 - 2\left(\frac{1}{8}\right) + 2\left(\frac{1}{8}\right)^2 \\ &= 1 - \frac{1}{4} + 2 \cdot \frac{1}{64} \\ &= 1 - \frac{1}{4} + \frac{1}{32} \\ &= \frac{25}{32} \end{aligned}$$

So

$$e^{-1/4} \approx \frac{25}{32}$$

2. (i) Use the Maclaurin series for $\sin x$ and $\cos x$ to work out the series expansion of

$$\sin 2x(\cos x - \cos 3x)$$

up to and including the term in x^5. **[4]**

(ii) Hence show that, in exact surd form, an approximation to the least positive root of the equation

$$2\sin 2x(\cos x - \cos 3x) = x$$

is

$$\sqrt{\frac{4-\sqrt{10}}{12}}$$

[3]

Solution

(i) Using the standard Maclaurin series,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots, \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

First expand $\sin 2x$:

$$\begin{aligned}\sin 2x &= 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \cdots \\ &= 2x - \frac{8x^3}{6} + \frac{32x^5}{120} + \cdots \\ &= 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 + \cdots\end{aligned}$$

Now expand $\cos x - \cos 3x$:

$$\begin{aligned}\cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots \\ \cos 3x &= 1 - \frac{(3x)^2}{2} + \frac{(3x)^4}{24} + \cdots \\ &= 1 - \frac{9x^2}{2} + \frac{81x^4}{24} + \cdots\end{aligned}$$

So

$$\begin{aligned}\cos x - \cos 3x &= \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) - \left(1 - \frac{9x^2}{2} + \frac{81x^4}{24}\right) + \cdots \\ &= 4x^2 - \frac{80x^4}{24} + \cdots \\ &= 4x^2 - \frac{10}{3}x^4 + \cdots\end{aligned}$$

Now multiply:

$$\sin 2x(\cos x - \cos 3x) = \left(2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 + \cdots\right)\left(4x^2 - \frac{10}{3}x^4 + \cdots\right)$$

Keeping terms up to x^5,

$$\begin{aligned}\sin 2x(\cos x - \cos 3x) &= (2x)(4x^2) + (2x)\left(-\frac{10}{3}x^4\right) + \left(-\frac{4}{3}x^3\right)(4x^2) + \cdots \\ &= 8x^3 - \frac{20}{3}x^5 - \frac{16}{3}x^5 + \cdots \\ &= 8x^3 - 12x^5 + \cdots\end{aligned}$$

Hence,

$$\sin 2x(\cos x - \cos 3x) = 8x^3 - 12x^5 + \cdots$$

(ii) From part (i),

$$2\sin 2x(\cos x - \cos 3x) \approx 2(8x^3 - 12x^5) = 16x^3 - 24x^5$$

So the equation

$$2\sin 2x(\cos x - \cos 3x) = x$$

becomes approximately

$$16x^3 - 24x^5 = x$$

Rearranging,

$$\begin{aligned} 16x^3 - 24x^5 - x &= 0 \\ x(16x^2 - 24x^4 - 1) &= 0 \\ x(24x^4 - 16x^2 + 1) &= 0 \end{aligned}$$

Since we want the least positive root, $x \neq 0$, so

$$24x^4 - 16x^2 + 1 = 0$$

Let $y = x^2$. Then

$$24y^2 - 16y + 1 = 0$$

Using the quadratic formula,

$$\begin{aligned} y &= \frac{16 \pm \sqrt{(-16)^2 - 4(24)(1)}}{2(24)} \\ &= \frac{16 \pm \sqrt{256 - 96}}{48} \\ &= \frac{16 \pm \sqrt{160}}{48} \\ &= \frac{16 \pm 4\sqrt{10}}{48} \\ &= \frac{4 \pm \sqrt{10}}{12} \end{aligned}$$

For the least positive root, take the smaller value:

$$x^2 \approx \frac{4 - \sqrt{10}}{12}$$

Hence

$$x \approx \sqrt{\frac{4 - \sqrt{10}}{12}}$$

So the least positive root is approximately

$$\sqrt{\frac{4 - \sqrt{10}}{12}} \approx 0.264$$

3. **(a)** Starting from the series given in the formulae booklet, show that the general term of the Maclaurin series for

$$2(1-\cos x) - x\sin x$$

is

$$(-1)^r \frac{2(r-1)}{(2r)!} x^{2r}$$

[4]

(b) Show that

$$\lim_{x\to 0}\left[\frac{2(1-\cos x) - x\sin x}{(1-\cos x)^2}\right] = \frac{1}{3}$$

[4]

Solution

(a) From the formula booklet,

$$\cos x = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{(2r)!}, \qquad \sin x = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r+1}}{(2r+1)!}$$

So

$$\begin{aligned} 2(1-\cos x) &= 2\left(1 - \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{(2r)!}\right) \\ &= 2\sum_{r=1}^{\infty} (-1)^{r+1} \frac{x^{2r}}{(2r)!} \end{aligned}$$

Also,

$$\begin{aligned} x\sin x &= x\sum_{r=0}^{\infty} (-1)^r \frac{x^{2r+1}}{(2r+1)!} \\ &= \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r+2}}{(2r+1)!} \end{aligned}$$

Re-index this sum by writing $r-1$ instead of r, so that it is also in powers of x^{2r}:

$$x\sin x = \sum_{r=1}^{\infty} (-1)^{r-1} \frac{x^{2r}}{(2r-1)!}$$

Hence

$$2(1-\cos x) - x\sin x = \sum_{r=1}^{\infty} \left(\frac{2(-1)^{r+1}}{(2r)!} - \frac{(-1)^{r-1}}{(2r-1)!}\right) x^{2r}$$

So the coefficient of x^{2r} is

$$\begin{aligned} \frac{2(-1)^{r+1}}{(2r)!} - \frac{(-1)^{r-1}}{(2r-1)!} &= (-1)^r \left(-\frac{2}{(2r)!} + \frac{1}{(2r-1)!}\right) \\ &= (-1)^r \left(-\frac{2}{(2r)!} + \frac{2r}{(2r)!}\right) \\ &= (-1)^r \frac{2(r-1)}{(2r)!} \end{aligned}$$

Therefore the general term of the Maclaurin series is

$$(-1)^r \frac{2(r-1)}{(2r)!} x^{2r}$$

as required.

(b) Using part (a), the first non-zero term in the numerator comes from $r = 2$:

$$\begin{aligned} 2(1-\cos x) - x\sin x &= \frac{2(2-1)}{4!}x^4 + \cdots \\ &= \frac{x^4}{12} + \cdots \end{aligned}$$

Now expand $1 - \cos x$:

$$\begin{aligned} 1 - \cos x &= \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots \\ &= \frac{x^2}{2} - \frac{x^4}{24} + \cdots \end{aligned}$$

Squaring gives

$$\begin{aligned} (1-\cos x)^2 &= \left(\frac{x^2}{2} - \frac{x^4}{24} + \cdots\right)^2 \\ &= \frac{x^4}{4} + \cdots \end{aligned}$$

Therefore

$$\frac{2(1-\cos x) - x\sin x}{(1-\cos x)^2} = \frac{\frac{x^4}{12} + \cdots}{\frac{x^4}{4} + \cdots}$$

and so, comparing the leading terms as $x \to 0$,

$$\lim_{x\to 0}\left[\frac{2(1-\cos x) - x\sin x}{(1-\cos x)^2}\right] = \frac{1/12}{1/4} = \frac{1}{3}$$

So the limit is $\frac{1}{3}$.

4. Let $f(x) = \ln(1 + \sin x)$.

(a) (i) Determine $f''(x)$. **[2]**

(ii) Determine the first two non-zero terms of the Maclaurin expansion for $f(x)$. **[3]**

(iii) By considering the first two non-zero terms of the Maclaurin expansion for $f(x)$, find an approximation to

$$\int_0^{\frac{2}{5}} f(x)\,\mathrm{d}x$$

Give your answer correct to 6 decimal places. **[2]**

Solution

(a) (i) Given

$$f(x) = \ln(1 + \sin x)$$

first differentiate using

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln u) = \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

so

$$f'(x) = \frac{\cos x}{1 + \sin x}$$

Differentiate again using the quotient rule:

$$\begin{aligned} f''(x) &= \frac{(-\sin x)(1 + \sin x) - \cos x(\cos x)}{(1 + \sin x)^2} \\ &= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2} \\ &= \frac{-\sin x - 1}{(1 + \sin x)^2} \\ &= -\frac{1}{1 + \sin x} \end{aligned}$$

Hence

$$f''(x) = -\frac{1}{1 + \sin x}$$

(ii) For the Maclaurin expansion, find the values at $x = 0$:

$$f(0) = \ln(1 + \sin 0) = \ln 1 = 0$$

$$f'(0) = \frac{\cos 0}{1 + \sin 0} = \frac{1}{1} = 1$$

$$f''(0) = -\frac{1}{1 + \sin 0} = -1$$

Using

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$$

gives

$$\begin{aligned} f(x) &= 0 + 1 \cdot x + \frac{-1}{2}x^2 + \cdots \\ &= x - \frac{x^2}{2} + \cdots \end{aligned}$$

So the first two non-zero terms are

$$x - \frac{x^2}{2}$$

(iii) Using the first two non-zero terms,

$$f(x) \approx x - \frac{x^2}{2}$$

Therefore

$$\begin{aligned}\int_0^{2/5} f(x)\,\mathrm{d}x &\approx \int_0^{2/5} \left(x - \frac{x^2}{2}\right)\,\mathrm{d}x \\ &= \left[\frac{x^2}{2} - \frac{x^3}{6}\right]_0^{2/5} \\ &= \frac{(2/5)^2}{2} - \frac{(2/5)^3}{6} \\ &= \frac{2}{25} - \frac{4}{375} \\ &= \frac{26}{375}\end{aligned}$$

Hence the required approximation is

$$\frac{26}{375} \approx 0.069333$$

correct to 6 decimal places.

5. **(a)** Given that

$$y = \cosh^{-1}\left(\frac{3+x}{2+x}\right)$$

show that

$$(x+2)\sqrt{5+2x}\,\frac{\mathrm{d}y}{\mathrm{d}x} + 1 = 0$$

[4]

(b) Hence find the first three terms in the Maclaurin series for $\cosh^{-1}\left(\frac{3+x}{2+x}\right)$ in the form

$$a\ln\left(\frac{3+\sqrt{5}}{2}\right) + bx + cx^2$$

where a, b and c are constants to be determined. **[5]**

Solution

(a) Let

$$y = \cosh^{-1}\left(\frac{x+3}{x+2}\right)$$

Then

$$\cosh y = \frac{x+3}{x+2}$$

Differentiate implicitly with respect to x:

$$\begin{aligned}\sinh y\,\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(x+2)(1)-(x+3)(1)}{(x+2)^2}\\ &= \frac{x+2-x-3}{(x+2)^2}\\ &= -\frac{1}{(x+2)^2}\end{aligned}$$

Now use

$$\sinh^2 y = \cosh^2 y - 1$$

So

$$\begin{aligned}\sinh^2 y &= \left(\frac{x+3}{x+2}\right)^2 - 1\\ &= \frac{(x+3)^2-(x+2)^2}{(x+2)^2}\\ &= \frac{x^2+6x+9-(x^2+4x+4)}{(x+2)^2}\\ &= \frac{2x+5}{(x+2)^2}\end{aligned}$$

Since $y = \cosh^{-1}(\cdots)$, we have $y \geq 0$, so $\sinh y \geq 0$. Also near $x = 0$, $x+2 > 0$. Hence

$$\sinh y = \frac{\sqrt{2x+5}}{x+2}$$

Substituting into the differentiated equation:

$$\frac{\sqrt{2x+5}}{x+2}\,\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{(x+2)^2}$$

so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{(x+2)\sqrt{2x+5}}$$

Therefore

$$(x+2)\sqrt{5+2x}\,\frac{\mathrm{d}y}{\mathrm{d}x}+1=0$$

So the given result is shown.

(b) From part (a),

$$(x+2)\sqrt{5+2x}\,\frac{\mathrm{d}y}{\mathrm{d}x}+1=0$$

First divide through by $\sqrt{5+2x}$:

$$(x+2)\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{1}{\sqrt{5+2x}}=0$$

Differentiate implicitly with respect to x:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left((x+2)\frac{\mathrm{d}y}{\mathrm{d}x}\right)+\frac{\mathrm{d}}{\mathrm{d}x}\left((5+2x)^{-1/2}\right)&=0\\(x+2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}-(5+2x)^{-3/2}&=0\end{aligned}$$

Now evaluate the required quantities at $x=0$.

First,

$$y(0)=\cosh^{-1}\left(\frac{3}{2}\right)$$

Using

$$\cosh^{-1}t=\ln\left(t+\sqrt{t^2-1}\right),$$

we get

$$\begin{aligned}y(0)&=\ln\left(\frac{3}{2}+\sqrt{\frac{9}{4}-1}\right)\\&=\ln\left(\frac{3}{2}+\sqrt{\frac{5}{4}}\right)\\&=\ln\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)\\&=\ln\left(\frac{3+\sqrt{5}}{2}\right)\end{aligned}$$

Also from part (a),

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{(x+2)\sqrt{5+2x}}$$

so

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0}=-\frac{1}{2\sqrt{5}}$$

Substitute $x=0$ into

$$\begin{aligned}(x+2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}-(5+2x)^{-3/2}&=0:\\2\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0}+\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0}-5^{-3/2}&=0\end{aligned}$$

Hence

$$\begin{aligned}2\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0}-\frac{1}{2\sqrt{5}}-\frac{1}{5\sqrt{5}}&=0\\2\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0}&=\frac{1}{2\sqrt{5}}+\frac{1}{5\sqrt{5}}\\&=\frac{7}{10\sqrt{5}}\end{aligned}$$

so

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0} = \frac{7}{20\sqrt{5}}$$

Therefore the Maclaurin series is

$$y = y(0) + \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} x + \frac{1}{2}\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0} x^2 + \cdots$$

Thus

$$\cosh^{-1}\left(\frac{3+x}{2+x}\right) = \ln\left(\frac{3+\sqrt{5}}{2}\right) - \frac{x}{2\sqrt{5}} + \frac{7x^2}{40\sqrt{5}} + \cdots$$

So

$$a = 1, \qquad b = -\frac{1}{2\sqrt{5}}, \qquad c = \frac{7}{40\sqrt{5}}$$

6.

$$y = (1 + \cosh x)^n \qquad n \geq 2$$

(a) (i) Show that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = n^2(1 + \cosh x)^n - n(2n-1)(1 + \cosh x)^{n-1}$$ **[4]**

(ii) Determine an expression for

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4}$$ **[2]**

(b) Hence determine the first three non-zero terms of the Maclaurin series for y, giving each coefficient in simplest form. **[2]**

Solution

(a) **(i)** Let

$$y = (1 + \cosh x)^n$$

Differentiate once:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = n(1 + \cosh x)^{n-1} \sinh x$$

Differentiate again using the product rule:

$$\begin{aligned} \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= n\left((n-1)(1 + \cosh x)^{n-2} \sinh^2 x + (1 + \cosh x)^{n-1} \cosh x\right) \\ &= n(n-1)(1 + \cosh x)^{n-2} \sinh^2 x + n(1 + \cosh x)^{n-1} \cosh x \end{aligned}$$

Now use

$$\sinh^2 x = \cosh^2 x - 1 = (\cosh x - 1)(\cosh x + 1)$$

Since $\cosh x + 1 = 1 + \cosh x$,

$$\begin{aligned} \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= n(n-1)(1 + \cosh x)^{n-2}(\cosh x - 1)(1 + \cosh x) + n(1 + \cosh x)^{n-1} \cosh x \\ &= n(n-1)(1 + \cosh x)^{n-1}(\cosh x - 1) + n(1 + \cosh x)^{n-1} \cosh x \\ &= n(1 + \cosh x)^{n-1} \left((n-1)(\cosh x - 1) + \cosh x\right) \\ &= n(1 + \cosh x)^{n-1} \left(n \cosh x - (n-1)\right) \end{aligned}$$

Write

$$n \cosh x = n(1 + \cosh x) - n$$

Then

$$n \cosh x - (n-1) = n(1 + \cosh x) - (2n-1)$$

So

$$\begin{aligned} \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= n(1 + \cosh x)^{n-1} \left(n(1 + \cosh x) - (2n-1)\right) \\ &= n^2(1 + \cosh x)^n - n(2n-1)(1 + \cosh x)^{n-1} \end{aligned}$$

Hence

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = n^2(1 + \cosh x)^n - n(2n-1)(1 + \cosh x)^{n-1}$$

(ii) From part (i),

$$y'' = n^2(1 + \cosh x)^n - n(2n-1)(1 + \cosh x)^{n-1}$$

Differentiate twice again. Using part (i) with powers n and $n-1$:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(1+\cosh x)^n = n^2(1+\cosh x)^n - n(2n-1)(1+\cosh x)^{n-1}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(1+\cosh x)^{n-1} = (n-1)^2(1+\cosh x)^{n-1} - (n-1)(2n-3)(1+\cosh x)^{n-2}$$

Therefore

$$\begin{aligned}\frac{\mathrm{d}^4y}{\mathrm{d}x^4} &= n^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}(1+\cosh x)^n - n(2n-1)\frac{\mathrm{d}^2}{\mathrm{d}x^2}(1+\cosh x)^{n-1}\\ &= n^2\left[n^2(1+\cosh x)^n - n(2n-1)(1+\cosh x)^{n-1}\right]\\ &\quad - n(2n-1)\left[(n-1)^2(1+\cosh x)^{n-1} - (n-1)(2n-3)(1+\cosh x)^{n-2}\right]\\ &= n^4(1+\cosh x)^n - n(2n-1)\left(n^2+(n-1)^2\right)(1+\cosh x)^{n-1}\\ &\quad + n(n-1)(2n-1)(2n-3)(1+\cosh x)^{n-2}\end{aligned}$$

Since

$$n^2+(n-1)^2 = 2n^2-2n+1$$

we get

$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = n^4(1+\cosh x)^n - n(2n-1)(2n^2-2n+1)(1+\cosh x)^{n-1} + n(n-1)(2n-1)(2n-3)(1+\cosh x)^{n-2}$$

(b) Since $\cosh x$ is an even function, $y=(1+\cosh x)^n$ is also even. So its Maclaurin series contains only even powers, and

$$y'(0)=0, \qquad y'''(0)=0$$

Also,

$$y(0) = (1+\cosh 0)^n = (1+1)^n = 2^n$$

From part (a)(i),

$$\begin{aligned}y''(0) &= n^2(2^n) - n(2n-1)(2^{n-1})\\ &= n2^{n-1}\left(2n-(2n-1)\right)\\ &= n2^{n-1}\end{aligned}$$

From part (a)(ii),

$$\begin{aligned}y''''(0) &= n^4 2^n - n(2n-1)(2n^2-2n+1)2^{n-1} + n(n-1)(2n-1)(2n-3)2^{n-2}\\ &= n2^{n-2}\left[4n^3 - 2(2n-1)(2n^2-2n+1) + (n-1)(2n-1)(2n-3)\right]\end{aligned}$$

Expand the bracket:

$$\begin{aligned}2(2n-1)(2n^2-2n+1) &= 8n^3-12n^2+8n-2\\ (n-1)(2n-1)(2n-3) &= 4n^3-12n^2+11n-3\end{aligned}$$

Hence

$$\begin{aligned}y''''(0) &= n2^{n-2}\left[4n^3-(8n^3-12n^2+8n-2)+(4n^3-12n^2+11n-3)\right]\\ &= n2^{n-2}(3n-1)\end{aligned}$$

Using

$$y = y(0) + \frac{y''(0)}{2!}x^2 + \frac{y''''(0)}{4!}x^4 + \cdots$$

gives

$$\begin{aligned}y &= 2^n + \frac{n2^{n-1}}{2}x^2 + \frac{n(3n-1)2^{n-2}}{24}x^4 + \cdots\\ &= 2^n + n2^{n-2}x^2 + \frac{n(3n-1)2^{n-2}}{24}x^4 + \cdots\end{aligned}$$

So the first three non-zero terms are

$$2^n + n2^{n-2}x^2 + \frac{n(3n-1)2^{n-2}}{24}x^4$$

7. (i) Prove that, if $y = \arctan x$, then
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2}$$
[3]

(ii) Find the Maclaurin series for $\arctan x$, up to and including the term in x^5. **[4]**

(iii) Use the result of part (ii) and the Maclaurin series for $\ln(1+t)$ to find the Maclaurin series for $\ln(1+\arctan x)$, up to and including the term in x^4. **[5]**

Solution

(i) Let
$$y = \arctan x$$
so equivalently
$$x = \tan y$$
Differentiate implicitly with respect to x:
$$\frac{\mathrm{d}}{\mathrm{d}x}(x) = \frac{\mathrm{d}}{\mathrm{d}x}(\tan y)$$
$$1 = \sec^2 y \frac{\mathrm{d}y}{\mathrm{d}x}$$
Hence
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sec^2 y}$$
Using the identity $\sec^2 y = 1 + \tan^2 y$, and $\tan y = x$, we get
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$
Therefore
$$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x) = \frac{1}{1+x^2}$$

(ii) From part (i),
$$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x) = \frac{1}{1+x^2}$$
Using the geometric series
$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots \qquad (|r| < 1)$$
with $r = -x^2$, we have
$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots \qquad (|x| < 1)$$
So
$$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x) = 1 - x^2 + x^4 - \cdots$$
Integrating term by term,
$$\arctan x = \int (1 - x^2 + x^4 - \cdots)\,\mathrm{d}x$$
$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + C$$
To find C, put $x = 0$:
$$\tan^{-1} 0 = 0$$
so $C = 0$.

Therefore the Maclaurin series is
$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

(iii) Let

$$t = \arctan x$$

Then from part (ii),

$$t = x - \frac{x^3}{3} + \cdots$$

We use the Maclaurin series

$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots$$

Now find the powers of t needed up to the term in x^4.

First,

$$t^2 = \left(x - \frac{x^3}{3} + \cdots\right)^2$$

$$t^2 = x^2 - \frac{2x^4}{3} + \cdots$$

Next,

$$t^3 = \left(x - \frac{x^3}{3} + \cdots\right)^3 = x^3 + \cdots$$

Also,

$$t^4 = \left(x - \frac{x^3}{3} + \cdots\right)^4 = x^4 + \cdots$$

Substitute into the logarithm series:

$$\begin{aligned} \ln(1+\arctan x) &= \ln(1+t) \\ &= t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \\ &= \left(x - \frac{x^3}{3} + \cdots\right) - \frac{1}{2}\left(x^2 - \frac{2x^4}{3} + \cdots\right) + \frac{1}{3}\left(x^3 + \cdots\right) - \frac{1}{4}\left(x^4 + \cdots\right) + \cdots \end{aligned}$$

Now simplify:

$$\begin{aligned} \ln(1+\arctan x) &= x - \frac{x^3}{3} - \frac{x^2}{2} + \frac{x^4}{3} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\ &= x - \frac{x^2}{2} + \left(\frac{1}{3} - \frac{1}{4}\right)x^4 + \cdots \\ &= x - \frac{x^2}{2} + \frac{x^4}{12} + \cdots \end{aligned}$$

Therefore

$$\ln(1+\arctan x) = x - \frac{x^2}{2} + \frac{x^4}{12} + \cdots$$

8. **(a)** The function f is defined as $f(x) = \operatorname{sech}^2 x$.

(i) Show that $f^{(4)}(0) = 16$. **[4]**

(ii) Hence find the first three non-zero terms of the Maclaurin series for $f(x) = \operatorname{sech}^2 x$. **[2]**

(b) Prove that

$$\lim_{x \to 0} \left(\frac{\operatorname{sech}^2 x - \cos(\sqrt{2}\,x)}{x^4} \right) = \frac{1}{2}$$ **[4]**

Solution

(a) (i) Let

$$f(x) = \operatorname{sech}^2 x$$

Using

$$\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

we get

$$f'(x) = 2 \operatorname{sech} x \big(-\operatorname{sech} x \tanh x\big) = -2 \operatorname{sech}^2 x \tanh x = -2f \tanh x$$

Differentiate again:

$$\begin{aligned} f''(x) &= -2f' \tanh x - 2f \operatorname{sech}^2 x \\ &= -2f' \tanh x - 2f^2 \end{aligned}$$

Since

$$\tanh^2 x = 1 - \operatorname{sech}^2 x = 1 - f$$

and $f' = -2f \tanh x$,

$$\begin{aligned} f''(x) &= -2(-2f \tanh x) \tanh x - 2f^2 \\ &= 4f \tanh^2 x - 2f^2 \\ &= 4f(1-f) - 2f^2 \\ &= 4f - 6f^2 \end{aligned}$$

Now evaluate at $x = 0$. Since $\operatorname{sech} 0 = 1$ and $\tanh 0 = 0$,

$$f(0) = 1$$

$$\begin{aligned} f'(0) &= -2f(0) \tanh 0 = 0 \\ f''(0) &= 4f(0) - 6f(0)^2 = 4 - 6 = -2 \end{aligned}$$

Differentiate $f'' = 4f - 6f^2$:

$$f''' = 4f' - 12ff'$$

so

$$f'''(0) = 4f'(0) - 12f(0)f'(0) = 0$$

Differentiate once more:

$$f^{(4)} = 4f'' - 12\big((f')^2 + ff''\big)$$

Hence

$$\begin{aligned} f^{(4)}(0) &= 4f''(0) - 12\big((f'(0))^2 + f(0)f''(0)\big) \\ &= 4(-2) - 12\big(0 + 1(-2)\big) \\ &= -8 + 24 \\ &= 16 \end{aligned}$$

Therefore

$$f^{(4)}(0) = 16$$

(ii) The Maclaurin series for $f(x)$ is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

Substituting the values found in part (i),

$$\begin{aligned} f(x) &= 1 + 0x + \frac{-2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{16}{4!}x^4 + \cdots \\ &= 1 - x^2 + \frac{16}{24}x^4 + \cdots \\ &= 1 - x^2 + \frac{2}{3}x^4 + \cdots \end{aligned}$$

So the first three non-zero terms are

$$\operatorname{sech}^2 x = 1 - x^2 + \frac{2}{3}x^4 + \cdots$$

(b) From part (a),

$$\operatorname{sech}^2 x = 1 - x^2 + \frac{2}{3}x^4 + \cdots$$

Also, from the standard Maclaurin series,

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots$$

Put $t = \sqrt{2}\,x$:

$$\begin{aligned} \cos(\sqrt{2}\,x) &= 1 - \frac{(\sqrt{2}\,x)^2}{2!} + \frac{(\sqrt{2}\,x)^4}{4!} + \cdots \\ &= 1 - \frac{2x^2}{2} + \frac{4x^4}{24} + \cdots \\ &= 1 - x^2 + \frac{1}{6}x^4 + \cdots \end{aligned}$$

Therefore

$$\begin{aligned} \operatorname{sech}^2 x - \cos(\sqrt{2}\,x) &= \left(1 - x^2 + \frac{2}{3}x^4 + \cdots\right) - \left(1 - x^2 + \frac{1}{6}x^4 + \cdots\right) \\ &= \left(\frac{2}{3} - \frac{1}{6}\right)x^4 + \cdots \\ &= \frac{1}{2}x^4 + \cdots \end{aligned}$$

So

$$\frac{\operatorname{sech}^2 x - \cos(\sqrt{2}\,x)}{x^4} = \frac{1}{2} + \text{terms involving positive powers of } x$$

As $x \to 0$, those remaining terms tend to 0. Hence

$$\lim_{x \to 0}\left(\frac{\operatorname{sech}^2 x - \cos(\sqrt{2}\,x)}{x^4}\right) = \frac{1}{2}$$

9. Using the Maclaurin series for $\sin x$, show that, for small values of x,

$$\frac{x}{\sin x} \approx 1 + ax^2 + bx^4 + cx^6$$

where the values of a, b and c are to be given in exact form. **[5]**

Solution

Using the Maclaurin series,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

so

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$$

Factor out x:

$$\sin x = x\left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + \cdots\right)$$

Hence

$$\frac{x}{\sin x} = \frac{1}{1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + \cdots}$$

We are told that, for small x,

$$\frac{x}{\sin x} \approx 1 + ax^2 + bx^4 + cx^6$$

So this expression must satisfy

$$(1 + ax^2 + bx^4 + cx^6)\left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040}\right) = 1 + \cdots$$

Now expand and collect powers of x:

$$\begin{aligned}&(1 + ax^2 + bx^4 + cx^6)\left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040}\right)\\ &= 1 + \left(a - \frac{1}{6}\right)x^2 + \left(b - \frac{a}{6} + \frac{1}{120}\right)x^4 + \left(c - \frac{b}{6} + \frac{a}{120} - \frac{1}{5040}\right)x^6 + \cdots\end{aligned}$$

For this to equal 1 up to and including the term in x^6, the coefficients of x^2, x^4 and x^6 must each be 0. So:

$$a - \frac{1}{6} = 0 \quad\Rightarrow\quad a = \frac{1}{6}$$

$$b - \frac{a}{6} + \frac{1}{120} = 0 \quad\Rightarrow\quad b = \frac{a}{6} - \frac{1}{120}$$

Substituting $a = \frac{1}{6}$,

$$b = \frac{1}{36} - \frac{1}{120} = \frac{7}{360}$$

$$c - \frac{b}{6} + \frac{a}{120} - \frac{1}{5040} = 0 \quad\Rightarrow\quad c = \frac{b}{6} - \frac{a}{120} + \frac{1}{5040}$$

Substituting $a = \frac{1}{6}$ and $b = \frac{7}{360}$,

$$\begin{aligned}c &= \frac{7}{2160} - \frac{1}{720} + \frac{1}{5040}\\ &= \frac{49 - 21 + 3}{15120}\\ &= \frac{31}{15120}\end{aligned}$$

Therefore,

$$a = \frac{1}{6}, \qquad b = \frac{7}{360}, \qquad c = \frac{31}{15120}$$

and

$$\frac{x}{\sin x} \approx 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \frac{31x^6}{15120}$$

for small values of x.

10. **(a)** Use the Maclaurin series expansion for $\ln(1+x)$ to show that the first three non-zero terms of the Maclaurin series expansion of

$$\ln\left(\frac{1+2x}{1-x}\right)$$

are

$$3x - \frac{3}{2}x^2 + 3x^3$$

[2]

(b) Sam attempts to use the series expansion found in part (a) to find an approximation for $\ln 6$.

Sam's incorrect working is shown below.

$$\begin{aligned} \text{Let } \frac{1+2x}{1-x} &= 6 \\ 1+2x &= 6-6x \\ 8x &= 5 \\ x &= \frac{5}{8} \end{aligned}$$

$$\begin{aligned} \text{So } \ln 6 &\approx 3\left(\frac{5}{8}\right) - \frac{3}{2}\left(\frac{5}{8}\right)^2 + 3\left(\frac{5}{8}\right)^3 \\ &\approx 2.02 \end{aligned}$$

Explain the error in Sam's working. **[2]**

(c) Use $x = \frac{1}{4}$ in the series expansion found in part (a) to find an approximation for $\ln 8$.

Fully justify your answer. **[3]**

Solution

(a) Using

$$\ln\left(\frac{1+2x}{1-x}\right) = \ln(1+2x) - \ln(1-x)$$

and the Maclaurin series

$$\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} + \cdots$$

we have, with $u = 2x$,

$$\ln(1+2x) = 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} + \cdots = 2x - 2x^2 + \frac{8}{3}x^3 + \cdots$$

Also, with $u = -x$,

$$\ln(1-x) = (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} + \cdots = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \cdots$$

So

$$\begin{aligned} \ln\left(\frac{1+2x}{1-x}\right) &= \left(2x - 2x^2 + \frac{8}{3}x^3 + \cdots\right) - \left(-x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \cdots\right) \\ &= 3x - \frac{3}{2}x^2 + 3x^3 + \cdots \end{aligned}$$

Hence the first three non-zero terms are

$$3x - \frac{3}{2}x^2 + 3x^3$$

(b) Sam's algebra for finding x is correct, since

$$\frac{1+2x}{1-x} = 6 \quad \Rightarrow \quad x = \frac{5}{8}$$

The error is that the series from part (a) is not valid for this value of x.

For $\ln(1+2x)$, the Maclaurin series for $\ln(1+u)$ requires

$$-1 < u \leq 1$$

so here

$$-1 < 2x \leq 1 \quad \Rightarrow \quad -\frac{1}{2} < x \leq \frac{1}{2}$$

Therefore the series for

$$\ln\left(\frac{1+2x}{1-x}\right)$$

is only valid for

$$-\frac{1}{2} < x \leq \frac{1}{2}$$

But

$$\frac{5}{8} > \frac{1}{2}$$

So Sam substituted a value of x outside the interval of convergence, and the approximation is invalid.

(c) Put $x = \frac{1}{4}$ into the expression inside the logarithm:

$$\frac{1+2x}{1-x} = \frac{1+2\left(\frac{1}{4}\right)}{1-\frac{1}{4}} = \frac{1+\frac{1}{2}}{\frac{3}{4}} = \frac{\frac{3}{2}}{\frac{3}{4}} = 2$$

So the series from part (a) gives an approximation for $\ln 2$:

$$\begin{aligned}\ln 2 &\approx 3\left(\frac{1}{4}\right) - \frac{3}{2}\left(\frac{1}{4}\right)^2 + 3\left(\frac{1}{4}\right)^3 \\ &= \frac{3}{4} - \frac{3}{32} + \frac{3}{64} \\ &= \frac{48-6+3}{64} \\ &= \frac{45}{64}\end{aligned}$$

Now

$$\ln 8 = \ln(2^3) = 3\ln 2$$

Hence

$$\ln 8 \approx 3 \times \frac{45}{64} = \frac{135}{64} = 2.109375$$

So

$$\ln 8 \approx \frac{135}{64} \approx 2.11$$

This is justified because $x = \frac{1}{4}$ lies in the valid interval

$$-\frac{1}{2} < x \leq \frac{1}{2}$$

Also, the next term in the series is found by continuing part (a):

$$\ln(1+2x) = 2x - 2x^2 + \frac{8}{3}x^3 - 4x^4 + \cdots$$

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

so

$$\ln\left(\frac{1+2x}{1-x}\right) = 3x - \frac{3}{2}x^2 + 3x^3 - \frac{15}{4}x^4 + \cdots$$

At $x = \frac{1}{4}$ this becomes

$$\ln 2 = \frac{3}{4} - \frac{3}{32} + \frac{3}{64} - \frac{15}{1024} + \cdots$$

This is an alternating series with decreasing terms, so the error is less than the first omitted term:

$$|\text{error in } \ln 2| < \frac{15}{1024}$$

Therefore

$$|\text{error in } \ln 8| < 3 \cdot \frac{15}{1024} = \frac{45}{1024} \approx 0.0439$$

Since the next omitted term is negative, $\frac{135}{64}$ is an overestimate.

Hence the approximation is

$$\ln 8 \approx \frac{135}{64} \approx 2.11$$

11. **(a)** By using an appropriate Maclaurin series prove that if $0 < x < 1$ then $-\ln(1-x) > x$. **[2]**

(b) Hence, by using a suitable substitution, deduce that $\ln t > 1 - \frac{1}{t}$ for $t > 1$. **[1]**

(c) Using the inequality in part (b), and by making a suitable choice for t, determine which is greater,

$$\left(\frac{5}{4}\right)^5 \quad \text{or} \quad e$$

[3]

Solution

(a) Using the Maclaurin series

$$\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \cdots \qquad (|u| < 1)$$

put $u = -x$. Then, for $0 < x < 1$,

$$\begin{aligned} \ln(1-x) &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \\ \therefore \quad -\ln(1-x) &= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots \end{aligned}$$

Since $0 < x < 1$, every term after the first is positive, so

$$-\ln(1-x) > x$$

Hence proved.

(b) Let

$$x = 1 - \frac{1}{t} = \frac{t-1}{t}$$

If $t > 1$, then $0 < x < 1$, so part (a) applies:

$$-\ln(1-x) > x$$

Substituting $x = 1 - \frac{1}{t}$ gives

$$\begin{aligned} -\ln\left(1 - \left(1 - \frac{1}{t}\right)\right) &> 1 - \frac{1}{t} \\ -\ln\left(\frac{1}{t}\right) &> 1 - \frac{1}{t} \\ \ln t &> 1 - \frac{1}{t} \end{aligned}$$

So, for $t > 1$,

$$\ln t > 1 - \frac{1}{t}$$

(c) Choose $t = \dfrac{5}{4}$ in part (b). Since $\dfrac{5}{4} > 1$,

$$\ln\left(\frac{5}{4}\right) > 1 - \frac{1}{5/4} = 1 - \frac{4}{5} = \frac{1}{5}$$

Multiply both sides by 5:

$$5\ln\left(\frac{5}{4}\right) > 1$$

But $1 = \ln e$, so

$$\ln\left(\left(\frac{5}{4}\right)^5\right) > \ln e$$

Since $\ln x$ is an increasing function,

$$\left(\frac{5}{4}\right)^5 > e$$

Therefore, $\left(\dfrac{5}{4}\right)^5$ is greater than e.

12. **(a)** Write down the first three non-zero terms of the Maclaurin series for $\ln\left(\dfrac{1+2x}{1-x}\right)$. **[2]**

(b) Use these three terms to show that

$$\ln\left(\frac{10}{7}\right) \approx \frac{183}{512}$$

[2]

(c) Dana uses the same first three terms of the series to approximate $\ln 10$ and gets an answer of 2.67, correct to 3 significant figures. However, $\ln 10 = 2.30$ correct to 3 significant figures.

Explain Dana's error. **[2]**

Solution

(a) Using

$$\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} + \cdots$$

we have

$$\ln\left(\frac{1+2x}{1-x}\right) = \ln(1+2x) - \ln(1-x)$$

Now expand each logarithm:

$$\ln(1+2x) = 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} + \cdots = 2x - 2x^2 + \frac{8x^3}{3} + \cdots$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} + \cdots$$

So

$$\begin{aligned}\ln\left(\frac{1+2x}{1-x}\right) &= \left(2x - 2x^2 + \frac{8x^3}{3}\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3}\right) + \cdots \\ &= 3x - \frac{3x^2}{2} + 3x^3 + \cdots\end{aligned}$$

The first three non-zero terms are

$$3x - \frac{3x^2}{2} + 3x^3$$

(b) To use the series for $\ln\left(\dfrac{10}{7}\right)$, first find x such that

$$\frac{1+2x}{1-x} = \frac{10}{7}$$

Hence

$$\begin{aligned}7(1+2x) &= 10(1-x) \\ 7 + 14x &= 10 - 10x \\ 24x = 3 &\implies x = \frac{1}{8}\end{aligned}$$

Substitute $x = \dfrac{1}{8}$ into the first three terms:

$$\begin{aligned}\ln\left(\frac{10}{7}\right) &\approx 3\left(\frac{1}{8}\right) - \frac{3}{2}\left(\frac{1}{8}\right)^2 + 3\left(\frac{1}{8}\right)^3 \\ &= \frac{3}{8} - \frac{3}{128} + \frac{3}{512} \\ &= \frac{192}{512} - \frac{12}{512} + \frac{3}{512} \\ &= \frac{183}{512}\end{aligned}$$

So

$$\ln\left(\frac{10}{7}\right) \approx \frac{183}{512}$$

(c) For $\ln 10$, Dana must have used

$$\frac{1+2x}{1-x} = 10$$

so

$$1 + 2x = 10 - 10x \implies 12x = 9 \implies x = \frac{3}{4}$$

However, the series in part (a) comes from the Maclaurin expansions of $\ln(1+2x)$ and $\ln(1-x)$.

The expansion for $\ln(1+2x)$ is only valid when

$$|2x| < 1 \implies |x| < \frac{1}{2}$$

Since $x = \frac{3}{4}$, this is outside the interval where the series converges.

So Dana's error was using the first three terms of the Maclaurin series at a value of x for which the series is not valid. That is why her answer 2.67 is not reliable.

13. **(a)** Use the Maclaurin series expansions for $\sin x$ and $\cos x$ to determine the series expansion of

$$\left(\frac{\sin(x/2)}{x/2}\right)\cos\left(\frac{x}{3}\right)$$

in ascending powers of x, up to and including the term in x^4.

Give each term in simplest form. **[3]**

(b) Use the answer to part (a) and calculus to find an approximation, to 5 decimal places, for

$$\int_{\pi/6}^{2\pi/3}\left(\frac{1}{x}\left(\frac{\sin(x/2)}{x/2}\right)\cos\left(\frac{x}{3}\right)\right)\,\mathrm{d}x$$ **[3]**

(c) Use the integration function on your calculator to evaluate

$$\int_{\pi/6}^{2\pi/3}\left(\frac{1}{x}\left(\frac{\sin(x/2)}{x/2}\right)\cos\left(\frac{x}{3}\right)\right)\,\mathrm{d}x$$

Give your answer to 5 decimal places. **[1]**

(d) Assuming that the calculator answer in part (c) is accurate to 5 decimal places, comment on the accuracy of the approximation found in part (b). **[1]**

Solution

(a) Using the standard Maclaurin series,

$$\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} + \cdots, \qquad \cos u = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} + \cdots$$

With $u = \frac{x}{2}$,

$$\begin{aligned}\sin\left(\frac{x}{2}\right) &= \frac{x}{2} - \frac{1}{3!}\left(\frac{x}{2}\right)^3 + \frac{1}{5!}\left(\frac{x}{2}\right)^5 + \cdots \\ &= \frac{x}{2} - \frac{x^3}{48} + \frac{x^5}{3840} + \cdots\end{aligned}$$

So

$$\frac{\sin(x/2)}{x/2} = 1 - \frac{x^2}{24} + \frac{x^4}{1920} + \cdots$$

Also, with $u = \frac{x}{3}$,

$$\begin{aligned}\cos\left(\frac{x}{3}\right) &= 1 - \frac{1}{2!}\left(\frac{x}{3}\right)^2 + \frac{1}{4!}\left(\frac{x}{3}\right)^4 + \cdots \\ &= 1 - \frac{x^2}{18} + \frac{x^4}{1944} + \cdots\end{aligned}$$

Now multiply the two series, keeping terms up to x^4:

$$\begin{aligned}\left(\frac{\sin(x/2)}{x/2}\right)\cos\left(\frac{x}{3}\right) &= \left(1 - \frac{x^2}{24} + \frac{x^4}{1920}\right)\left(1 - \frac{x^2}{18} + \frac{x^4}{1944}\right) \\ &= 1 - \left(\frac{1}{24} + \frac{1}{18}\right)x^2 + \left(\frac{1}{1920} + \frac{1}{1944} + \frac{1}{24}\cdot\frac{1}{18}\right)x^4 + \cdots \\ &= 1 - \frac{7x^2}{72} + \frac{521x^4}{155520} + \cdots\end{aligned}$$

Hence

$$\left(\frac{\sin(x/2)}{x/2}\right)\cos\left(\frac{x}{3}\right) = 1 - \frac{7x^2}{72} + \frac{521x^4}{155520} + O(x^6)$$

(b) Using part (a),

$$\frac{1}{x}\left(\frac{\sin(x/2)}{x/2}\right)\cos\left(\frac{x}{3}\right) \approx \frac{1}{x}\left(1-\frac{7x^2}{72}+\frac{521x^4}{155520}\right) = \frac{1}{x}-\frac{7x}{72}+\frac{521x^3}{155520}$$

Therefore

$$\begin{aligned} I &= \int_{\pi/6}^{2\pi/3}\left(\frac{1}{x}\left(\frac{\sin(x/2)}{x/2}\right)\cos\left(\frac{x}{3}\right)\right)\,\mathrm{d}x \\ &\approx \int_{\pi/6}^{2\pi/3}\left(\frac{1}{x}-\frac{7x}{72}+\frac{521x^3}{155520}\right)\,\mathrm{d}x \\ &= \left[\ln x-\frac{7x^2}{144}+\frac{521x^4}{622080}\right]_{\pi/6}^{2\pi/3} \end{aligned}$$

Now simplify:

$$\begin{aligned} \ln\left(\frac{2\pi}{3}\right)-\ln\left(\frac{\pi}{6}\right) &= \ln\left(\frac{2\pi/3}{\pi/6}\right) \\ &= \ln 4 \end{aligned}$$

$$\begin{aligned} -\frac{7}{144}\left[\left(\frac{2\pi}{3}\right)^2-\left(\frac{\pi}{6}\right)^2\right] &= -\frac{7}{144}\left(\frac{4\pi^2}{9}-\frac{\pi^2}{36}\right) \\ &= -\frac{7}{144}\cdot\frac{15\pi^2}{36} \\ &= -\frac{35\pi^2}{1728} \end{aligned}$$

$$\begin{aligned} \frac{521}{622080}\left[\left(\frac{2\pi}{3}\right)^4-\left(\frac{\pi}{6}\right)^4\right] &= \frac{521}{622080}\left(\frac{16\pi^4}{81}-\frac{\pi^4}{1296}\right) \\ &= \frac{521}{622080}\cdot\frac{255\pi^4}{1296} \\ &= \frac{8857\pi^4}{53747712} \end{aligned}$$

So

$$I \approx \ln 4-\frac{35\pi^2}{1728}+\frac{8857\pi^4}{53747712}$$

Hence

$$I \approx 1.20244$$

So the approximation is 1.20244 to 5 decimal places.

(c) Using the calculator integration function,

$$\int_{\pi/6}^{2\pi/3}\left(\frac{1}{x}\left(\frac{\sin(x/2)}{x/2}\right)\cos\left(\frac{x}{3}\right)\right)\,\mathrm{d}x \approx 1.20169$$

So the calculator value is 1.20169.

(d) Comparing the two answers,

$$1.20244-1.20169=0.00075$$

So the approximation in part (b) is an overestimate by about 0.00075.

It is accurate to 3 decimal places, but not to 4 decimal places.

14. **(a)** Find and simplify the first five terms in the Maclaurin series for e^x. **[2]**

(b) Hence, or otherwise, write down the first five terms in the Maclaurin series for e^{-x}. **[1]**

(c) Use your answers, together with the identity $\cosh x + 1 = 2\cosh^2\left(\frac{x}{2}\right)$, to show that the Maclaurin series for $\cosh^2\left(\frac{x}{2}\right)$ is

$$a + bx^2 + cx^4 + \cdots$$

where a, b and c are rational numbers to be determined. **[3]**

Solution

(a) Using the standard Maclaurin series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

so the first five terms, simplified, are

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

(b) Replace x by $-x$ in the series for e^x:

$$e^{-x} = 1 + (-x) + \frac{(-x)^2}{2} + \frac{(-x)^3}{6} + \frac{(-x)^4}{24} + \cdots$$

Hence

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

(c) Using

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

and the expansions from parts (a) and (b),

$$\begin{aligned} e^x + e^{-x} &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right) + \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right) \\ &= 2 + x^2 + \frac{x^4}{12} + \cdots \end{aligned}$$

So

$$\cosh x = \frac{2 + x^2 + \frac{x^4}{12} + \cdots}{2} = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \cdots$$

Now use the identity

$$\cosh x + 1 = 2\cosh^2\left(\frac{x}{2}\right)$$

Substituting the series for $\cosh x$:

$$2\cosh^2\left(\frac{x}{2}\right) = \left(1 + \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right) + 1 = 2 + \frac{x^2}{2} + \frac{x^4}{24} + \cdots$$

Divide by 2:

$$\cosh^2\left(\frac{x}{2}\right) = 1 + \frac{x^2}{4} + \frac{x^4}{48} + \cdots$$

Therefore,

$$a = 1, \qquad b = \frac{1}{4}, \qquad c = \frac{1}{48}$$

15. **(i)** Use the Maclaurin series for $\ln(1+x)$ and $\ln(1-x)$ to obtain the first three non-zero terms in the Maclaurin series for

$$\ln\left(\frac{(1+x)^2}{1-2x}\right)$$

State the range of validity of this series. **[4]**

(ii) Find the value of x in the range of validity for which

$$\frac{(1+x)^2}{1-2x} = 2$$

Hence find an approximation to $\ln 2$, giving your answer to three decimal places. **[4]**

Solution

(i) Using

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

and

$$\ln(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)$$

we have

$$\ln\left(\frac{(1+x)^2}{1-2x}\right) = 2\ln(1+x) - \ln(1-2x)$$

Now substitute into each series.

For $2\ln(1+x)$,

$$2\ln(1+x) = 2\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right) = 2x - x^2 + \frac{2x^3}{3} - \cdots$$

For $-\ln(1-2x)$,

$$\ln(1-2x) = -\left(2x + \frac{(2x)^2}{2} + \frac{(2x)^3}{3} + \cdots\right)$$

so

$$-\ln(1-2x) = 2x + \frac{(2x)^2}{2} + \frac{(2x)^3}{3} + \cdots = 2x + 2x^2 + \frac{8x^3}{3} + \cdots$$

Hence

$$\begin{aligned}\ln\left(\frac{(1+x)^2}{1-2x}\right) &= \left(2x - x^2 + \frac{2x^3}{3}\right) + \left(2x + 2x^2 + \frac{8x^3}{3}\right) + \cdots \\ &= 4x + x^2 + \frac{10x^3}{3} + \cdots\end{aligned}$$

So the first three non-zero terms are

$$4x + x^2 + \frac{10x^3}{3}$$

For validity:

$\ln(1+x)$ requires $-1 < x \le 1$, and $\ln(1-2x)$ requires $-\frac{1}{2} \le x < \frac{1}{2}$.

So the combined series is valid for

$$|x| < \frac{1}{2}$$

and it also converges at $x = -\frac{1}{2}$.

(ii) We solve

$$\frac{(1+x)^2}{1-2x} = 2$$

So

$$\begin{aligned}(1+x)^2 &= 2(1-2x)\\ 1+2x+x^2 &= 2-4x\\ x^2+6x-1 &= 0\end{aligned}$$

Using the quadratic formula,

$$x = \frac{-6 \pm \sqrt{36+4}}{2} = \frac{-6 \pm \sqrt{40}}{2} = -3 \pm \sqrt{10}$$

The root in the range of validity is

$$x = \sqrt{10} - 3$$

since $\sqrt{10} - 3 \approx 0.162$, while $-3 - \sqrt{10}$ is not in the valid range.

Now

$$\ln 2 = \ln\left(\frac{(1+x)^2}{1-2x}\right)$$

so using the series from part (i),

$$\ln 2 \approx 4x + x^2 + \frac{10x^3}{3}$$

Substituting $x = \sqrt{10} - 3$,

$$\ln 2 \approx 4(\sqrt{10}-3) + (\sqrt{10}-3)^2 + \frac{10}{3}(\sqrt{10}-3)^3$$

This simplifies to

$$\ln 2 \approx \frac{364\sqrt{10} - 1149}{3}$$

Hence

$$\ln 2 \approx 0.689688\ldots$$

So, to three decimal places,

$$\ln 2 \approx 0.690$$

16. Find the Maclaurin series for

$$e^{e^{x^2}-x}$$

up to and including the term in x^2. **[4]**

Solution

Using the standard Maclaurin expansion

$$e^t = 1 + t + \frac{t^2}{2} + \cdots$$

we first expand e^{x^2}:

$$e^{x^2} = 1 + x^2 + \cdots$$

So the exponent becomes

$$e^{x^2} - x = 1 - x + x^2 + \cdots$$

Hence

$$e^{e^{x^2}-x} = e^{\,1-x+x^2+\cdots} = e \cdot e^{-x+x^2+\cdots}$$

Let

$$u = -x + x^2 + \cdots$$

Then

$$e^u = 1 + u + \frac{u^2}{2} + \cdots$$

Now

$$u^2 = (-x + x^2 + \cdots)^2 = x^2 - 2x^3 + \cdots$$

so, up to the term in x^2,

$$u^2 = x^2 + \cdots$$

Therefore

$$\begin{aligned} e^u &= 1 + \left(-x + x^2\right) + \frac{1}{2}\left(x^2\right) + \cdots \\ &= 1 - x + \frac{3}{2}x^2 + \cdots \end{aligned}$$

Multiplying by e,

$$e^{e^{x^2}-x} = e\left(1 - x + \frac{3}{2}x^2 + \cdots\right)$$

So the Maclaurin series up to and including the term in x^2 is

$$e^{e^{x^2}-x} = e - ex + \frac{3e}{2}x^2 + \cdots$$

17. **(a)** Using the definitions of $\sinh x$ and $\cosh x$, together with the Maclaurin series expansion of e^x, find the first three non-zero terms in the Maclaurin series expansion of $x\cosh x - \sinh x$. **[3]**

(b) Hence, by replacing x with $\mathrm{i}x$ in your answer to part (a), find the first three non-zero terms in the Maclaurin series expansion of $x\cos x - \sin x$. **[3]**

Solution

(a) Using

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$$

we have

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \frac{x^7}{7!} + \cdots$$

From the definitions,

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

So

$$\begin{aligned}\cosh x &= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\right]\\ &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\\ &= 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \cdots\end{aligned}$$

and

$$\begin{aligned}\sinh x &= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\right]\\ &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\\ &= x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots\end{aligned}$$

Therefore

$$\begin{aligned}x\cosh x - \sinh x &= x\left(1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \cdots\right) - \left(x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots\right)\\ &= \left(x + \frac{x^3}{2} + \frac{x^5}{24} + \frac{x^7}{720} + \cdots\right) - \left(x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots\right)\\ &= \left(\frac{1}{2} - \frac{1}{6}\right)x^3 + \left(\frac{1}{24} - \frac{1}{120}\right)x^5 + \left(\frac{1}{720} - \frac{1}{5040}\right)x^7 + \cdots\\ &= \frac{x^3}{3} + \frac{x^5}{30} + \frac{x^7}{840} + \cdots\end{aligned}$$

Hence the first three non-zero terms are

$$x\cosh x - \sinh x = \frac{x^3}{3} + \frac{x^5}{30} + \frac{x^7}{840} + \cdots$$

(b) Replace x by $\mathrm{i}x$ in the result from part (a):

$$(\mathrm{i}x)\cosh(\mathrm{i}x) - \sinh(\mathrm{i}x) = \frac{(\mathrm{i}x)^3}{3} + \frac{(\mathrm{i}x)^5}{30} + \frac{(\mathrm{i}x)^7}{840} + \cdots$$

Using

$$\cosh(\mathrm{i}x) = \cos x, \qquad \sinh(\mathrm{i}x) = \mathrm{i}\sin x$$

gives

$$\mathrm{i}(x\cos x-\sin x)=\frac{(\mathrm{i}x)^3}{3}+\frac{(\mathrm{i}x)^5}{30}+\frac{(\mathrm{i}x)^7}{840}+\cdots$$

Now simplify the powers of i:

$$(\mathrm{i}x)^3=-\mathrm{i}x^3,\qquad (\mathrm{i}x)^5=\mathrm{i}x^5,\qquad (\mathrm{i}x)^7=-\mathrm{i}x^7$$

So

$$\mathrm{i}(x\cos x-\sin x)=-\frac{\mathrm{i}x^3}{3}+\frac{\mathrm{i}x^5}{30}-\frac{\mathrm{i}x^7}{840}+\cdots$$

Dividing through by i,

$$x\cos x-\sin x=-\frac{x^3}{3}+\frac{x^5}{30}-\frac{x^7}{840}+\cdots$$

Hence the first three non-zero terms are

$$x\cos x-\sin x=-\frac{x^3}{3}+\frac{x^5}{30}-\frac{x^7}{840}+\cdots$$