Questions

1. **(a)** (i) Given that $f(x) = e^{-2x}$, find $f'(x)$ and $f''(x)$. **[2]**

(ii) Hence, find the first three terms of the Maclaurin series for e^{-2x}. **[2]**

(b) Hence, using a suitable value of x, show that

$$e^{-1/4} \approx \frac{25}{32}$$

. **[2]**

2. (i) Use the Maclaurin series for $\sin x$ and $\cos x$ to work out the series expansion of

$$\sin 2x(\cos x - \cos 3x)$$

up to and including the term in x^5. **[4]**

(ii) Hence show that, in exact surd form, an approximation to the least positive root of the equation

$$2\sin 2x(\cos x - \cos 3x) = x$$

is

$$\sqrt{\frac{4-\sqrt{10}}{12}}$$ **[3]**

3. (a) Starting from the series given in the formulae booklet, show that the general term of the Maclaurin series for

$$2(1-\cos x) - x\sin x$$

is

$$(-1)^r \frac{2(r-1)}{(2r)!} x^{2r}$$

[4]

(b) Show that

$$\lim_{x\to 0}\left[\frac{2(1-\cos x) - x\sin x}{(1-\cos x)^2}\right] = \frac{1}{3}$$

[4]

4. Let $f(x) = \ln(1 + \sin x)$.

(a) (i) Determine $f''(x)$. **[2]**

(ii) Determine the first two non-zero terms of the Maclaurin expansion for $f(x)$. **[3]**

(iii) By considering the first two non-zero terms of the Maclaurin expansion for $f(x)$, find an approximation to

$$\int_0^{\frac{2}{5}} f(x)\,\mathrm{d}x$$

Give your answer correct to 6 decimal places. **[2]**

5. **(a)** Given that

$$y = \cosh^{-1}\left(\frac{3+x}{2+x}\right)$$

show that

$$(x+2)\sqrt{5+2x}\,\frac{\mathrm{d}y}{\mathrm{d}x} + 1 = 0$$

[4]

(b) Hence find the first three terms in the Maclaurin series for $\cosh^{-1}\left(\frac{3+x}{2+x}\right)$ in the form

$$a\ln\left(\frac{3+\sqrt{5}}{2}\right) + bx + cx^2$$

where a, b and c are constants to be determined. **[5]**

6.

$$y = (1 + \cosh x)^n \qquad n \geq 2$$

(a) (i) Show that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = n^2(1 + \cosh x)^n - n(2n - 1)(1 + \cosh x)^{n-1}$$ **[4]**

(ii) Determine an expression for

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4}$$ **[2]**

(b) Hence determine the first three non-zero terms of the Maclaurin series for y, giving each coefficient in simplest form. **[2]**

7. (i) Prove that, if $y = \arctan x$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2}$$ **[3]**

(ii) Find the Maclaurin series for $\arctan x$, up to and including the term in x^5. **[4]**

(iii) Use the result of part (ii) and the Maclaurin series for $\ln(1+t)$ to find the Maclaurin series for $\ln(1+\arctan x)$, up to and including the term in x^4. **[5]**

8. **(a)** The function f is defined as $f(x) = \operatorname{sech}^2 x$.

(i) Show that $f^{(4)}(0) = 16$. **[4]**

(ii) Hence find the first three non-zero terms of the Maclaurin series for $f(x) = \operatorname{sech}^2 x$. **[2]**

(b) Prove that

$$\lim_{x \to 0} \left(\frac{\operatorname{sech}^2 x - \cos(\sqrt{2}\, x)}{x^4} \right) = \frac{1}{2}$$ **[4]**

9. Using the Maclaurin series for $\sin x$, show that, for small values of x,

$$\frac{x}{\sin x} \approx 1 + ax^2 + bx^4 + cx^6$$

where the values of a, b and c are to be given in exact form. **[5]**

10. **(a)** Use the Maclaurin series expansion for $\ln(1+x)$ to show that the first three non-zero terms of the Maclaurin series expansion of

$$\ln\left(\frac{1+2x}{1-x}\right)$$

are

$$3x - \frac{3}{2}x^2 + 3x^3$$

[2]

(b) Sam attempts to use the series expansion found in part (a) to find an approximation for $\ln 6$.

Sam's incorrect working is shown below.

$$\text{Let } \frac{1+2x}{1-x} = 6$$
$$1+2x = 6-6x$$
$$8x = 5$$
$$x = \frac{5}{8}$$

$$\text{So } \ln 6 \approx 3\left(\frac{5}{8}\right) - \frac{3}{2}\left(\frac{5}{8}\right)^2 + 3\left(\frac{5}{8}\right)^3$$
$$\approx 2.02$$

Explain the error in Sam's working. **[2]**

(c) Use $x = \frac{1}{4}$ in the series expansion found in part (a) to find an approximation for $\ln 8$.

Fully justify your answer. **[3]**

11. **(a)** By using an appropriate Maclaurin series prove that if $0 < x < 1$ then $-\ln(1-x) > x$. **[2]**

(b) Hence, by using a suitable substitution, deduce that $\ln t > 1 - \frac{1}{t}$ for $t > 1$. **[1]**

(c) Using the inequality in part (b), and by making a suitable choice for t, determine which is greater,

$$\left(\frac{5}{4}\right)^5 \quad \text{or} \quad e$$

[3]

12. **(a)** Write down the first three non-zero terms of the Maclaurin series for $\ln\left(\dfrac{1+2x}{1-x}\right)$. **[2]**

(b) Use these three terms to show that

$$\ln\left(\frac{10}{7}\right) \approx \frac{183}{512}$$

[2]

(c) Dana uses the same first three terms of the series to approximate $\ln 10$ and gets an answer of 2.67, correct to 3 significant figures. However, $\ln 10 = 2.30$ correct to 3 significant figures.

Explain Dana's error. **[2]**

13. (a) Use the Maclaurin series expansions for $\sin x$ and $\cos x$ to determine the series expansion of

$$\left(\frac{\sin(x/2)}{x/2}\right)\cos\left(\frac{x}{3}\right)$$

in ascending powers of x, up to and including the term in x^4.

Give each term in simplest form. **[3]**

(b) Use the answer to part (a) and calculus to find an approximation, to 5 decimal places, for

$$\int_{\pi/6}^{2\pi/3}\left(\frac{1}{x}\left(\frac{\sin(x/2)}{x/2}\right)\cos\left(\frac{x}{3}\right)\right)\,dx$$ **[3]**

(c) Use the integration function on your calculator to evaluate

$$\int_{\pi/6}^{2\pi/3}\left(\frac{1}{x}\left(\frac{\sin(x/2)}{x/2}\right)\cos\left(\frac{x}{3}\right)\right)\,dx$$

Give your answer to 5 decimal places. **[1]**

(d) Assuming that the calculator answer in part (c) is accurate to 5 decimal places, comment on the accuracy of the approximation found in part (b). **[1]**

14. **(a)** Find and simplify the first five terms in the Maclaurin series for e^x. **[2]**

(b) Hence, or otherwise, write down the first five terms in the Maclaurin series for e^{-x}. **[1]**

(c) Use your answers, together with the identity $\cosh x + 1 = 2\cosh^2\left(\frac{x}{2}\right)$, to show that the Maclaurin series for $\cosh^2\left(\frac{x}{2}\right)$ is

$$a + bx^2 + cx^4 + \cdots$$

where a, b and c are rational numbers to be determined. **[3]**

15. **(i)** Use the Maclaurin series for $\ln(1+x)$ and $\ln(1-x)$ to obtain the first three non-zero terms in the Maclaurin series for

$$\ln\left(\frac{(1+x)^2}{1-2x}\right)$$

State the range of validity of this series. **[4]**

(ii) Find the value of x in the range of validity for which

$$\frac{(1+x)^2}{1-2x} = 2$$

Hence find an approximation to $\ln 2$, giving your answer to three decimal places. **[4]**

16. Find the Maclaurin series for

$$e^{e^{x^2}-x}$$

up to and including the term in x^2. **[4]**

17. **(a)** Using the definitions of $\sinh x$ and $\cosh x$, together with the Maclaurin series expansion of e^x, find the first three non-zero terms in the Maclaurin series expansion of $x\cosh x - \sinh x$. **[3]**

(b) Hence, by replacing x with $\mathrm{i}x$ in your answer to part (a), find the first three non-zero terms in the Maclaurin series expansion of $x\cos x - \sin x$. **[3]**