Questions

1. Find the exact value of the improper integral

$$\int_0^4 \frac{1}{\sqrt{x(4-x)}}\,\mathrm{d}x$$ **[5]**

2. Show that

$$\int_0^3 \frac{1}{x^2 - 2x + 5}\,\mathrm{d}x = \frac{1}{2}\arctan(3)$$

[5]

3. (i) Prove, from definitions involving exponentials, that

$$1 - \tanh^2 u = \operatorname{sech}^2 u$$ **[3]**

(ii) Prove that, for $|y| < 1$,

$$\operatorname{artanh} y = \frac{1}{2} \ln \left(\frac{1+y}{1-y} \right)$$ **[4]**

(iii) Use the substitution $x = 4 \tanh u$ to show that, for $|x| < 4$,

$$\int \frac{1}{16 - x^2} \, dx = \frac{1}{4} \operatorname{artanh} \frac{x}{4} + c$$

Hence deduce that

$$\int \frac{1}{16 - x^2} \, dx = \frac{1}{8} \ln \left(\frac{4+x}{4-x} \right) + c$$

where c is an arbitrary constant. **[6]**

(iv) By first expressing $-t^2 + 6t + 7$ in completed square form, show that

$$\int_1^5 \frac{1}{-t^2 + 6t + 7} \, dt = \frac{1}{4} \ln 3$$ **[5]**

4. (i) For $x > 5$, find

$$\int \frac{1}{\sqrt{x^2 - 6x + 5}}\,\mathrm{d}x$$

expressing your answer first in terms of arcosh and hence in logarithmic form. **[3]**

(ii) Use integration by substitution with $x - 3 = 2\cosh u$ to show that, for $x > 5$,

$$\int \sqrt{x^2 - 6x + 5}\,\mathrm{d}x = 2\left(\frac{x-3}{2}\sqrt{\frac{(x-3)^2}{4} - 1} - \ln\left|\frac{x-3}{2} + \sqrt{\frac{(x-3)^2}{4} - 1}\right|\right) + c$$

where c is an arbitrary constant. **[7]**

5.

$$g(x) = \frac{2x^2 - 3x + 31}{x^3 - 3x^2 + 9x + 13}$$

(a) Express $g(x)$ in the form

$$\frac{P}{x+1} + \frac{Q}{x^2 - 4x + 13}$$

where P and Q are constants to be found. **[3]**

(b) Find $\int g(x)\,\mathrm{d}x$, giving your answer in the form

$$A \ln|x+1| + B \arctan\left(\frac{x-2}{3}\right) + c$$

where A and B are constants to be found. **[4]**

(c) Hence show that $\int_0^\infty g(x)\,\mathrm{d}x$ diverges. **[2]**

6. Show that

$$\int_1^4 \frac{1}{\sqrt{10+8x-2x^2}}\,\mathrm{d}x = \frac{1}{\sqrt{2}}\left(\arcsin\left(\frac{2}{3}\right) + \arcsin\left(\frac{1}{3}\right)\right)$$

[6]

7. **(a)** Explain why

$$\int_0^\infty \frac{1}{x^2 - 6x + 18}\,\mathrm{d}x$$

is an improper integral. **[1]**

(b) Show that

$$\int_0^\infty \frac{1}{x^2 - 6x + 18}\,\mathrm{d}x = k\pi$$

where k is a constant to be determined. **[4]**

8. Evaluate

$$\int_0^4 \frac{1}{x^2 - 4x + 8}\,\mathrm{d}x$$

giving your answer as an exact multiple of π. **[8]**

9.

$$f(x) = \frac{6x+1}{4x^2 - 8x + 13}$$

(a) Find $\int f(x)\,\mathrm{d}x$, giving your answer in the form

$$A\ln(4x^2 - 8x + 13) + B\arctan\left(\frac{2x-2}{3}\right) + c$$

where c is an arbitrary constant and A and B are constants to be found. **[4]**

(b) Hence show that the mean value of $f(x)$ over the interval $\left[1, \frac{5}{2}\right]$ is

$$\frac{1}{2}\ln 2 + \frac{7\pi}{36}$$ **[3]**

(c) Hence write down the mean value of $3 - 2f(x)$ over the interval $\left[1, \frac{5}{2}\right]$. **[1]**

10.

$$g(x) = \frac{4x - 1}{x^2 - 4x + 8}$$

Show that

$$\int g(x)\,\mathrm{d}x = A\arctan\left(\frac{x-2}{2}\right) + B\ln(x^2 - 4x + 8) + c$$

where c is an arbitrary constant and A and B are constants to be found. **[5]**

11. You are given that $y = \operatorname{arcosh} x$, where $x \geq 1$.

(i) Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{x^2 - 1}}$$ **[4]**

(ii) Show, by integrating the result in part (i), that

$$y = \ln\left(x + \sqrt{x^2 - 1}\right)$$ **[4]**

(iii) Hence show that

$$\int_{15/8}^{5/2} \frac{1}{\sqrt{4x^2 - 9}}\,\mathrm{d}x = \frac{1}{2}\left(\operatorname{arcosh}\frac{5}{3} - \operatorname{arcosh}\frac{5}{4}\right)$$

Express this answer in logarithmic form. **[4]**

12. Determine each of the following integrals.

(i)

$$\int \frac{1}{9x^2 + 12x + 20} \, \mathrm{d}x$$ **[4]**

(ii)

$$\int \frac{1}{\sqrt{45 + 12x - 9x^2}} \, \mathrm{d}x$$ **[4]**

13. Without using a calculator, find

(a)

$$\int_{-1/5}^{1} \frac{1}{25x^2 - 20x + 13}\,\mathrm{d}x$$

giving your answer as a multiple of π **[5]**

(b)

$$\int_{-1/5}^{1} \frac{1}{\sqrt{25x^2 - 20x + 13}}\,\mathrm{d}x$$

giving your answer in the form $p\ln(q + r\sqrt{2})$, where p, q and r are rational numbers to be found. **[7]**

14. (i) Given that $\tanh y = x$, where $|x| < 1$, show that

$$y = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$$

and hence that

$$\operatorname{artanh} x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$$

[6]

(ii) Use your result in part (i), or otherwise, to find

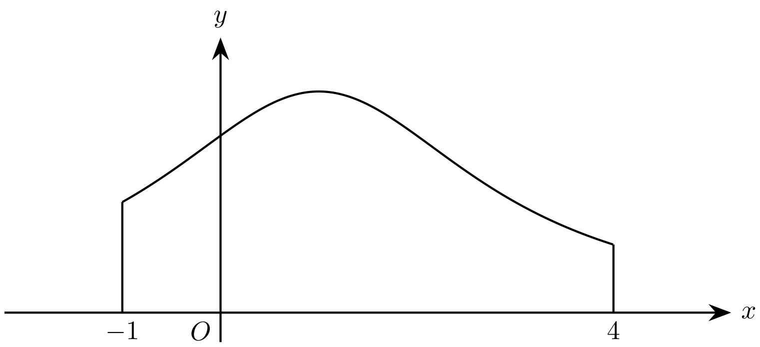
$$\int_{1/5}^{2/5} \frac{1}{4-9x^2} \, \mathrm{d}x$$

expressing your answer in an exact logarithmic form. **[6]**

15. Show that

$$\int \frac{x^2 + 14x + 21}{(x-1)(x^2 + 4x + 13)}\,\mathrm{d}x = A\ln|x-1| + B\ln(x^2 + 4x + 13) + D\arctan\left(\frac{x+2}{3}\right) + c$$

where A, B and D are constants to be found. **[7]**



16. The diagram shows the cross-section of a solid wooden beam.
Each unit on the axes represents 6 cm. The curved upper boundary of the beam is modelled by the equation

$$y = \frac{9}{(x-1)^2 + 4}$$

The cross-section is bounded by the curve, the x-axis and the lines $x = -1$ and $x = 4$.
Assuming that the beam is a prism of length 240 cm, find, correct to 3 significant figures, the volume of the beam. **[6]**

17. Find

$$\int_1^e \frac{1}{x\sqrt{4-(\ln x)^2}}\,\mathrm{d}x$$

expressing your answer in terms of π. **[8]**