CORE PURE / FURTHER CALCULUS

Volumes of Revolution

Mark scheme 15 questions · 110 marks

EXAM BOARDS Edexcel 9FM0 · AQA 7367 · OCR A H245 · OCR MEI H645 · CIE 9231

Questions

Question	Marks by part	Marks	Page
Question 1	(a) 2 (b) 4	6	4
Question 2	(a) 2 (b) 5	7	5
Question 3		5	6
Question 4		5	7
Question 5	(a) 2 (b) 5 (c) 3	10	8
Question 6		8	9
Question 7		5	10
Question 8		7	11
Question 9		6	12
Question 10	(a) 5 (b) 2 (c) 2	9	13
Question 11		6	14
Question 12		7	15
Question 13	(a) 3 (b) 6	9	16
Question 14	(a) 4 (b) 7	11	17
Question 15	(a) 3 (b) 6	9	18
	Total	**110**	

Reading the mark scheme

These rules apply throughout, unless a note says otherwise.

Marks

Every mark is worth one, so **M1 A1** on a row is two marks. Only marks coded ft follow through.

Code	Meaning
M1	Method: a correct method, applied to this question. A slip doesn't cost it if the method is clear. A formula must be used, not just quoted.
dM1	Dependent method: as M1, but only if the method mark before it was earned, unless its note names another.
A1	Accuracy: a correct answer or step. It needs the method mark it follows.
A1ft	As A1, following through from an earlier answer that the note names.
A1 AG	The last mark of a part whose answer is printed in the question: complete working, with no errors.
B1	A correct value, fact or statement. It needs no method mark.
B1ft	As B1, following through from an earlier answer that the note names.
E1	A written explanation, interpretation or conclusion. Concluding a proof by induction needs every earlier mark.

Abbreviations

ft and AG qualify a code in the Marks column. The others appear in the notes. A grey line gives the rule where the name alone doesn't.

ft	follow through Correct work from the student's own earlier wrong value earns the mark.	**AG**	answer given The answer is printed in the question, so every step to it must be shown.
cao	correct answer only No follow-through from an earlier error.	**cso**	correct solution only No error anywhere in the part, not just in its answer.
oe	or equivalent	**awrt**	answers which round to
soi	seen or implied Earned if written, or if later working shows it was used.	**isw**	ignore subsequent working A correct answer, once reached, isn't spoiled by later working.
SC	special case Credit, as the note says, for work outside the main method.	**their**	the student's own earlier value
condone	a slip that doesn't cost the mark	[ ]	working the mark doesn't need
s.f.	significant figures	**d.p.**	decimal places

Reading the scheme

- Each row is a line of working, with the marks it earns beside it. Where a row carries two marks, each note starts with its code.
- [***n***] ends each part with its total.
- A shaded *Alternative* is a second method for the same total. Mark an answer by the method it follows.
- Any other valid method earns the equivalent marks, unless the question names a method or says *hence*.
- A sketch earns one B1 for each feature named below its figure.

Accuracy and answers

- Unless a question says otherwise, give answers to 3 significant figures, and angles in degrees to 1 decimal place. Angles in radians take 3 significant figures.
- A final answer with more figures is accepted if it rounds to the scheme's value; one with fewer is not. An answer spoiled by rounding too early loses its accuracy mark.
- Where a question asks for an exact answer, a decimal loses the final accuracy mark. Otherwise either is accepted.
- Wrong or missing units cost nothing unless the question asks for them.
- An omitted $\mathrm{d}\theta$, $\mathrm{d}x$ or $\mathrm{d}u$ in an integral is not penalised where the variable of integration is clear from the integrand and the limits. It is penalised where it is genuinely ambiguous: a substitution part-way through, two variables in play, or limits in the wrong variable.
- A correct answer with no working earns full marks, unless the question needs the working: show that, prove, hence, a method it names, or an exact answer. A wrong answer with no working earns nothing. Calculator answers are treated the same way.
- A correct answer, once reached, isn't spoiled by later working (isw).

Value of g

- Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question states a value, which is then the only one accepted. Answers that use $g = 9.8$ may be given to 2 or 3 significant figures.
- Where a question doesn't state g, an answer worked with $g = 10$ throughout earns the same marks, keeping any value printed in the question. The worked solutions list the final answers with $g = 10$.

Question	Scheme	Marks	Notes
1(a)	$\frac{\mathrm{d}}{\mathrm{d}x}\ln(\sec x+\tan x)=\frac{\sec x\tan x+\sec^2 x}{\sec x+\tan x}$	**M1**	Divides a derivative of $\sec x+\tan x$ of the form $p\sec x\tan x+q\sec^2 x$ by $\sec x+\tan x$, oe in sines and cosines.
	$=\frac{\sec x(\tan x+\sec x)}{\sec x+\tan x}=\sec x$	**A1 AG**	Correct derivative simplified without error to $\sec x$; combined steps accepted.
		[2]	
1(b)	$V=\pi\int_0^{\pi/3}\sec x\,\mathrm{d}x$	**B1**	Needs the π; limits may be implied by their substitution.
	$=\pi\left[\ln(\sec x+\tan x)\right]_0^{\pi/3}$	**M1**	Obtains $\ln(\sec x+\tan x)$, or another correct integral of $\sec x$. The π need not be seen.
	$=\pi\left(\ln(2+\sqrt{3})-\ln 1\right)$	**dM1**	Substitutes both limits and subtracts, evaluating sec and tan exactly. Condone the zero lower-limit term unwritten.
	$V=\pi\ln(2+\sqrt{3})$	**A1**	oe exact, such as $-\pi\ln(2-\sqrt{3})$.
		[4]	
	Total for Question 1	**6**	

Question	Scheme	Marks	Notes
2(a)	$\sinh(2\ln 2) = \frac{e^{2\ln 2} - e^{-2\ln 2}}{2}$ $e^{2\ln 2} = e^{\ln 4} = 4$	**M1**	Uses the exponential definition of sinh with $2\ln 2$, or a double-angle identity with $\ln 2$, and evaluates at least one exponential.
	$\sinh(2\ln 2) = \frac{4 - \frac{1}{4}}{2} = \frac{15}{8}$	**A1 AG**	Correct substituted expression leading without error to $\frac{15}{8}$; combined steps accepted.
		[2]	
2(b)	$V = \pi \int_0^{2\ln 2} (\cosh x - 1)\,\mathrm{d}x$	**B1**	Correct integral with limits 0 and $2\ln 2$, oe such as a cylinder of volume $2\pi \ln 2$ subtracted. Needs the π.
	$\int (\cosh x - 1)\,\mathrm{d}x = \sinh x - x$	**M1 A1**	**M1** Integrates at least one term correctly, such as $\cosh x \to \sinh x$. **A1** With or without π.
	$\pi\left[\sinh x - x\right]_0^{2\ln 2} = \pi\left(\frac{15}{8} - 2\ln 2 - 0\right)$	**dM1**	Applies their limits to their integral of the form $p\sinh x + qx$, evaluating $\sinh(2\ln 2)$ by (a) or otherwise.
	$V = \pi\left(\frac{15}{8} - 2\ln 2\right)$	**A1**	Any exact equivalent, such as $\frac{15\pi}{8} - \pi\ln 4$.
		[5]	
Total for Question 2		**7**	

Question	Scheme	Marks	Notes
3	$\pi \int_0^k \frac{k-x}{k+x}\,\mathrm{d}x = 1$	**B1**	Needs π, y^2 as the integrand and limits 0 and k; the $= 1$ may be seen later.
	$\frac{k-x}{k+x} = -1 + \frac{2k}{k+x}$ $\Rightarrow \int \ldots \,\mathrm{d}x = \ldots$	**M1**	Writes the integrand as a constant plus $\frac{c}{k+x}$ and integrates to the form $px + q\ln(k+x)$. Condone sign and coefficient slips.
	$[-x + 2k\ln(k+x)]_0^k$	**A1**	Accept $\ln\lvert k+x\rvert$ or any equivalent, such as $2k\ln(k+x) - x + c$.
	$\pi\left(-k + 2k\ln 2k - 2k\ln k\right) = 1$ $\Rightarrow \pi k(2\ln 2 - 1) = 1 \Rightarrow k = \ldots$	**dM1**	Substitutes both limits into their integral, subtracts, and combines the logarithms to a constant multiple of k.
	$k = \frac{1}{\pi(2\ln 2 - 1)}$	**A1**	Or any exact equivalent, such as $\frac{1}{\pi(\ln 4 - 1)}$.
	Total for Question 3	**5**	

Question	Scheme	Marks	Notes
4	$\pi \int_0^k \frac{x^2}{k^2+x^2}\,\mathrm{d}x = 1$	**B1**	Needs the π and limits 0 and k; the equation with 1 may appear later.
	$\frac{x^2}{k^2+x^2} = 1 - \frac{k^2}{k^2+x^2}$ $\Rightarrow \int \ldots \,\mathrm{d}x = x - \ldots \arctan \frac{x}{k}$	**M1**	Writes the integrand as $1 - \frac{p}{k^2+x^2}$ and integrates to $x \pm q \arctan \frac{x}{k}$, or substitutes $x = k\tan\theta$ and integrates $\tan^2\theta$ via $\sec^2\theta - 1$. Condone coefficient slips.
	$\left[x - k\arctan\frac{x}{k}\right]_0^k$	**A1**	oe, such as $k(\tan\theta - \theta)$ after $x = k\tan\theta$.
	$\pi\left(k - k\arctan 1\right) = 1 \Rightarrow k = \ldots$	**dM1**	Substitutes both limits, in the variable used, into their integral, equates π times the result to 1 and solves for k.
	$k = \frac{4}{\pi(4-\pi)}$	**A1**	Any exact equivalent, such as $\frac{1}{\pi\left(1-\frac{\pi}{4}\right)}$.
	Total for Question 4	**5**	

Question	Scheme	Marks	Notes
5(a)	$\int \sin^2\theta\,\mathrm{d}\theta = \int \frac{1}{2}(1-\cos 2\theta)\,\mathrm{d}\theta$ $= \ldots$	**M1**	Uses $\sin^2\theta = \frac{1}{2}(1 \pm \cos 2\theta)$ and integrates to the form $p\theta + q\sin 2\theta$. Condone sign and coefficient slips.
	$= \frac{\theta}{2} - \frac{\sin 2\theta}{4} + c$	**A1**	oe, with $+c$.
		[2]	
5(b)	$\frac{\mathrm{d}y}{\mathrm{d}\theta} = -\operatorname{cosec}^2\theta$	**B1**	
	$V = \pi\int(6\sin^2\theta)^2 \times$ "$-\operatorname{cosec}^2\theta$" $\mathrm{d}\theta$	**M1**	Uses $\pi\int x^2\frac{\mathrm{d}y}{\mathrm{d}\theta}\,\mathrm{d}\theta$ with $x = 6\sin^2\theta$ and their $\frac{\mathrm{d}y}{\mathrm{d}\theta}$ of the form $\pm p\operatorname{cosec}^2\theta$. Limits not needed.
	$= 36\pi\int -\sin^2\theta\,\mathrm{d}\theta$	**A1**	oe. Limits not needed.
	$y = \sqrt{3} \Rightarrow \theta = \frac{\pi}{6}$ $y = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{3}$	**B1**	Both, matched to their y values; may be seen as limits.
	$V = 36\pi\int_{\pi/6}^{\pi/3}\sin^2\theta\,\mathrm{d}\theta$ $k = 36\pi,\ a = \frac{\pi}{6},\ b = \frac{\pi}{3}$	**A1**	cso. Accept $k = -36\pi$, $a = \frac{\pi}{3}$, $b = \frac{\pi}{6}$.
		[5]	
5(c)	$V = 36\pi\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{\pi/6}^{\pi/3}$	**M1**	Substitutes their limits from (b) into their integral from (a), of the form $p\theta + q\sin 2\theta$, and subtracts.
	$= 36\pi\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8} - \frac{\pi}{12} + \frac{\sqrt{3}}{8}\right)$	**dM1**	Uses exact values of $\sin\frac{2\pi}{3}$ and $\sin\frac{\pi}{3}$, or their equality, to reach an exact expression.
	$= 3\pi^2$	**A1**	cso. $p = 3$ alone scores A0.
		[3]	
	Total for Question 5	**10**	

Question	Scheme	Marks	Notes
6	$t^2 = y + 3 \Rightarrow x = \frac{9}{(y+3)^2}$	**B1**	oe. Implied by a correct integrand in t.
	$V = \pi \int_0^a \frac{81}{(y+3)^4}\,\mathrm{d}y$	**M1 A1**	**M1** Uses $\pi \int x^2\,\mathrm{d}y$ with their x in terms of y, or $\pi \int x^2 \frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t$. Needs the π; limits not needed. **A1** Correct integral with limits 0 and a, or $162\pi \int_{\sqrt{3}}^{\sqrt{a+3}} t^{-7}\,\mathrm{d}t$.
	$\pi\left[-\frac{27}{(y+3)^3}\right]_0^a$	**M1 A1**	**M1** Integrates to the form $k(y+3)^{-3}$, or kt^{-6} in t. Condone coefficient slips. **A1** oe, such as $-27\pi t^{-6}$.
	$\pi\left(1 - \frac{27}{(a+3)^3}\right) = \frac{7\pi}{8}$	**dM1**	Substitutes both limits into their integral, subtracts the right way round and sets equal to $\frac{7\pi}{8}$.
	$(a+3)^3 = 216$	**A1**	oe, such as $t^6 = 216$.
	$a = 3$	**A1**	
	Total for Question 6	**8**	

Question	Scheme	Marks	Notes
7	$(V =)\pi \int_0^8 \left(4 - \frac{x+4}{x+1}\right) \mathrm{d}x$	**B1**	Needs π and both limits. Accept oe, such as $32\pi - \pi \int_0^8 \frac{x+4}{x+1} \mathrm{d}x$.
	$4 - \frac{x+4}{x+1} = \frac{3x}{x+1} = 3 - \frac{3}{x+1}$ $\Rightarrow \int \left(3 - \frac{3}{x+1}\right) \mathrm{d}x = \ldots$	**M1**	Writes their integrand as a constant plus a multiple of $\frac{1}{x+1}$, by division or substitution, and integrates to the form $px + q\ln(x+1)$. Condone coefficient slips.
	$\pi \left[3x - 3\ln(x+1)\right]_0^8$	**A1**	Correct integral of the correct integrand, oe such as $x + 3\ln(x+1)$ for $\frac{x+4}{x+1}$.
	$\pi\left((24 - 3\ln 9) - (0 - 3\ln 1)\right)$	**dM1**	Substitutes both limits (1 and 9 if in $u = x+1$) into their integral and subtracts.
	$= 24\pi - 6\pi \ln 3$	**A1**	Any exact form, such as $3\pi(8 - \ln 9)$.
	Total for Question 7	**5**	

Question	Scheme	Marks	Notes
8	$\frac{e^a-e^{-a}}{2}=2 \Rightarrow e^{2a}-4e^a-1=0$ $\Rightarrow e^a=\ldots$	**M1**	Writes $\sinh a=2$ in exponentials and forms a three-term quadratic in e^a, or uses $\operatorname{arsinh} x=\ln\left(x+\sqrt{x^2+1}\right)$. Condone sign slips.
	$e^a=2+\sqrt{5} \Rightarrow a=\ln(2+\sqrt{5})$	**A1**	A0 if $2-\sqrt{5}$ is retained.
	$V=\pi\int_0^a\left(4-\sinh^2 x\right)\mathrm{d}x$	**B1**	Accept the cylinder $4\pi a$ minus $\pi\int_0^a \sinh^2 x\,\mathrm{d}x$. Upper limit a, $\operatorname{arsinh} 2$ or their logarithm. Needs the π.
	$\pi\int_0^a\left(4-\frac{1}{2}(\cosh 2x-1)\right)\mathrm{d}x$ $=\pi\left[\frac{9}{2}x-\frac{1}{4}\sinh 2x\right]_0^a$	**M1 A1**	**M1** Uses an identity for $\sinh^2 x$, or exponentials, and integrates to the form $px+q\sinh 2x$ or its exponential equivalent. Condone sign and coefficient slips. **A1** Correct integral, oe; π not needed.
	$\cosh a=\sqrt{5} \Rightarrow \sinh 2a=2(2)(\sqrt{5})=4\sqrt{5}$ $V=\pi\left(\frac{9}{2}a-\frac{1}{4}(4\sqrt{5})\right)$	**dM1**	Substitutes both limits and evaluates $\sinh 2a$ exactly, from $\sinh a=2$ with $\cosh^2 a=1+\sinh^2 a$, or from their e^a.
	$V=\pi\left(\frac{9}{2}\ln(2+\sqrt{5})-\sqrt{5}\right)$	**A1**	Exact oe, with a logarithm; $\operatorname{arsinh} 2$ in place of the logarithm scores A0.
	Total for Question 8	**7**	

Question	Scheme	Marks	Notes
9	$V = \pi \int x^2 \frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t,\ \frac{\mathrm{d}y}{\mathrm{d}t} = 16t$ $= \pi \int \left(12t(1-t)^2\right)^2 (16t)\,\mathrm{d}t$	**M1**	Uses $\pi \int x^2 \frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t$ with their $\frac{\mathrm{d}y}{\mathrm{d}t}$ and x substituted. Condone a slip in $\frac{\mathrm{d}y}{\mathrm{d}t}$.
	$V = 2304\pi \int_0^1 t^3(1-t)^4\,\mathrm{d}t$	**A1**	Correct integral with limits 0 and 1, oe unsimplified.
	$2304\pi \int_0^1 \left(t^3 - 4t^4 + 6t^5 - 4t^6 + t^7\right)\mathrm{d}t$ $= 2304\pi \left[\frac{t^4}{4} - \frac{4t^5}{5} + t^6 - \frac{4t^7}{7} + \frac{t^8}{8}\right]_0^1$	**M1 A1**	**M1** Expands their integrand into a polynomial, or substitutes $u = 1-t$, and integrates, raising at least one power by 1. **A1** Correct integrated terms, oe.
	$= 2304\pi \left(\frac{1}{4} - \frac{4}{5} + 1 - \frac{4}{7} + \frac{1}{8}\right)$	**dM1**	Substitutes $t = 1$ and $t = 0$, or limits in their variable, into their integrated expression.
	$V = \frac{288\pi}{35}$ $[\approx 25.9]$	**A1**	$\frac{288\pi}{35}$ oe, or awrt 25.9.
	Total for Question 9	**6**	

Question	Scheme	Marks	Notes
10(a)	$V = \pi \int_0^{12} e^{\frac{x}{4}} \left(1 - \frac{x}{12}\right) \mathrm{d}x$	**B1**	The π and the limits may appear later, soi.
	$\int e^{\frac{x}{4}} \left(1 - \frac{x}{12}\right) \mathrm{d}x$ $= 4e^{\frac{x}{4}} \left(1 - \frac{x}{12}\right) + \frac{1}{3} \int e^{\frac{x}{4}} \mathrm{d}x$	**M1**	Uses parts, differentiating the linear factor, on their $e^{kx} \left(1 - \frac{x}{12}\right)$ or on xe^{kx} after expanding, to reach $pe^{kx}(\ldots) \pm q \int e^{kx} \mathrm{d}x$. Condone sign and coefficient slips.
	$= 4e^{\frac{x}{4}} \left(1 - \frac{x}{12}\right) + \frac{4}{3} e^{\frac{x}{4}}$	**A1**	Correct integral, oe such as $e^{\frac{x}{4}} \left(\frac{16}{3} - \frac{x}{3}\right)$. The π and the limits are not needed here.
	$\left[4e^{\frac{x}{4}} \left(1 - \frac{x}{12}\right) + \frac{4}{3} e^{\frac{x}{4}}\right]_0^{12}$ $= \left(0 + \frac{4}{3} e^3\right) - \left(4 + \frac{4}{3}\right)$	**dM1**	Substitutes 12 and 0 into their integral of the form $e^{kx}(p + qx)$ oe and subtracts. Need not be simplified.
	$V = \frac{4\pi}{3} \left(e^3 - 4\right)$ cm^3	**A1**	Any exact equivalent, such as $\pi \left(\frac{4}{3} e^3 - \frac{16}{3}\right)$.
		[5]	
10(b)	"$\frac{4\pi}{3} \left(e^3 - 4\right)$" $- \pi \times 0.4^2 \times 5 = \ldots$	**M1**	Subtracts $\pi \times 0.4^2 \times 5$, oe such as 0.8π or awrt 2.51, from their answer to (a).
	64.9 cm^3	**A1**	awrt 64.9
		[2]	
10(c)	$0.72 \times$ "64.87" $= \ldots$ [≈ 46.7]	**M1**	Uses the density with their volume from (b) or with 46, for the predicted mass, the actual volume [awrt 63.9] or the implied density [awrt 0.709].
	46.7 g is close to 46 g [about 1.5% over] so the refined model is good	**A1ft**	ft their volume from (b). Correct mass for their volume [awrt 46.7], or the matching volume or density, compared with the given value. Concludes good if within 5%, otherwise not.
		[2]	
	Total for Question 10	**9**	

Question	Scheme	Marks	Notes
11	$V = \pi \int x^2 \frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t = \pi \int \frac{4}{(1+t)^2} \cdot 2t\,\mathrm{d}t$	**M1**	Forms $\pi \int x^2 \frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t$ with their x^2 in t and $\frac{\mathrm{d}y}{\mathrm{d}t} = kt$. Needs the π; limits not needed.
	$V = 8\pi \int \frac{t}{(1+t)^2}\,\mathrm{d}t$	**A1**	Correct integrand oe; limits need not be seen.
	$\frac{t}{(1+t)^2} = \frac{1}{1+t} - \frac{1}{(1+t)^2}$ $\Rightarrow \int \ldots\,\mathrm{d}t = \ldots$	**M1**	Integrates by any valid method (partial fractions, $u = 1+t$, parts) to the form $p\ln(1+t) + \frac{q}{1+t}$ oe. Condone coefficient slips.
	$8\pi\left[\ln(1+t) + \frac{1}{1+t}\right]$	**A1**	Correct integral oe, such as $\ln(1+t) - \frac{t}{1+t}$ or in u; the 8π need not be present.
	$t^2+1 = 2 \Rightarrow t = 1$, $t^2+1 = 5 \Rightarrow t = 2$ $8\pi\left[\ldots\right]_1^2 = 8\pi\left(\ln 3 + \frac{1}{3} - \ln 2 - \frac{1}{2}\right)$	**dM1**	Finds $t = 1$ and $t = 2$ from $y = 2$ and $y = 5$ and substitutes both into their integral, subtracting. Equivalent limits after a substitution accepted.
	$V = 8\pi \ln\frac{3}{2} - \frac{4\pi}{3}$ $[\approx 6.00]$	**A1**	oe, or awrt 6.00.
	Total for Question 11	**6**	

Question	Scheme	Marks	Notes
12	$\sin x = 2\sin x \cos x \Rightarrow \cos x = \ldots$	**M1**	Uses $\sin 2x = 2\sin x \cos x$ and solves for $\cos x$, by factorising or dividing by $\sin x$. Condone not stating $x = 0$.
	$x = \frac{\pi}{3}$	**A1**	
	$V = \pi \int_0^{\pi/3} \left(\sin^2 2x - \sin^2 x\right) \mathrm{d}x$	**B1**	oe, such as the difference of two volume integrals. Needs the π and the difference of the squares.
	$\sin^2 2x - \sin^2 x = \frac{1}{2}(\cos 2x - \cos 4x)$ $\Rightarrow \pi \int \frac{1}{2}(\cos 2x - \cos 4x)\,\mathrm{d}x = \ldots$	**M1**	Uses a double-angle identity on both squares and integrates to the form $px + q\sin 2x + r\sin 4x$. Condone sign and coefficient slips.
	$\pi \left[\frac{\sin 2x}{4} - \frac{\sin 4x}{8}\right]_0^{\pi/3}$	**A1**	oe, with or without the π.
	$\pi\left(\frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{16}\right)$	**dM1**	Substitutes $\frac{\pi}{3}$ (or their upper limit) and 0 into their integral of that form and subtracts, using exact values.
	$V = \frac{3\sqrt{3}\pi}{16}$	**A1**	oe in terms of π, or awrt 0.325π. A0 for awrt 1.02 alone.
	Total for Question 12	**7**	

Question	Scheme	Marks	Notes
13(a)	$\beta = \operatorname{arcosh}(1+2\alpha) \Rightarrow \cosh\beta = 1+2\alpha$	**M1**	Applies cosh to the curve's equation at P, or equates the log form of $\operatorname{arcosh}(1+2\alpha)$ to $\ln(2+\sqrt{3})$.
	$\operatorname{arcosh} 2 = \ln\left(2+\sqrt{2^2-1}\right) = \ln(2+\sqrt{3})$ $\Rightarrow \cosh\beta = 2$	**M1**	Uses $\operatorname{arcosh} u = \ln(u+\sqrt{u^2-1})$ with $u=2$, or substitutes β into the exponential form of cosh. Not for $\cosh\beta = 2$ from a calculator.
	$1+2\alpha = 2 \Rightarrow \alpha = \frac{1}{2}$	**A1 AG**	Correct exact working leading without error to $\alpha = \frac{1}{2}$. Working back from $\alpha = \frac{1}{2}$ earns nothing.
		[3]	
13(b)	$\cosh y = 1+2x \Rightarrow x = \frac{\cosh y - 1}{2}$	**B1**	
	$V = \pi\int_0^\beta \left(\frac{\cosh y-1}{2}\right)^2 \mathrm{d}y$	**B1**	Needs π, x^2 as their function of y, and limits 0 and β oe.
	$(\cosh y - 1)^2 = \frac{1}{2}(\cosh 2y + 1) - 2\cosh y + 1$ $\Rightarrow \int(\cosh y - 1)^2 \,\mathrm{d}y = \ldots$	**M1**	Expands, uses a double-angle identity or exponentials, and integrates to the form $p\sinh 2y + q\sinh y + ry$ or its exponential equivalent. Condone sign and coefficient slips.
	$V = \frac{\pi}{4}\left[\frac{1}{4}\sinh 2y - 2\sinh y + \frac{3}{2}y\right]_0^\beta$	**A1**	Correct integral oe, including exponential forms. The factor $\frac{\pi}{4}$ may be absent here.
	$\cosh\beta = 2 \Rightarrow \sinh\beta = \sqrt{3},\ \sinh 2\beta = 4\sqrt{3}$ $V = \frac{\pi}{4}\left(\frac{1}{4}(4\sqrt{3}) - 2\sqrt{3} + \frac{3}{2}\beta\right)$	**dM1**	Substitutes the limits, with $\sinh\beta$ and $\sinh 2\beta$ (or $e^{\pm\beta}$, $e^{\pm 2\beta}$) found exactly. Condone slips. Not for decimals.
	$V = \frac{\pi}{8}\left(3\ln(2+\sqrt{3}) - 2\sqrt{3}\right)$	**A1**	Any exact equivalent; $\ln(2+\sqrt{3})$ may be left as β or $\operatorname{arcosh} 2$.
		[6]	
Total for Question 13		**9**	

Question	Scheme	Marks	Notes
14(a)	$x = \sin\theta - \frac{1}{3}\left(3\sin\theta - 4\sin^3\theta\right)$	**M1**	Uses a triple-angle identity for $\sin 3\theta$, or any valid expansion, to write x in terms of $\sin\theta$ alone. Condone sign slips.
	$x = \frac{4}{3}\sin^3\theta$	**A1**	
	$\cos\theta = 1 - y \Rightarrow \sin^2\theta = 1 - (1-y)^2$ $x^2 = \text{“}\frac{16}{9}\text{”}\sin^6\theta$ $= \text{“}\frac{16}{9}\text{”}\left(1-(1-y)^2\right)^3$	**M1**	Uses $\sin^2\theta + \cos^2\theta = 1$ with $\cos\theta = 1 - y$, and substitutes into their x^2 of the form $k\sin^6\theta$.
	$x^2 = \frac{16}{9}\left(2y - y^2\right)^3 = \frac{16}{9}y^3(2-y)^3$	**A1 AG**	Correct substituted equation leading without error to the printed result; combined steps accepted.
		[4]	
14(b)	$V = \pi\int_0^2 \frac{16}{9}y^3(2-y)^3\,\mathrm{d}y$	**B1**	Needs the π, the integrand from (a) and limits 0 and 2.
	$\int_0^2 \left(8y^3 - 12y^4 + 6y^5 - y^6\right)\mathrm{d}y$ $= \left[2y^4 - \frac{12}{5}y^5 + y^6 - \frac{1}{7}y^7\right]_0^2$	**M1 A1**	**M1** Integrates $y^3(2-y)^3$ by expanding or any valid method, raising the power of at least one term by 1. Condone expansion slips. **A1** Correct integral, with or without the factor $\frac{16\pi}{9}$.
	$\frac{16\pi}{9}\left(32 - \frac{384}{5} + 64 - \frac{128}{7}\right)$	**dM1**	Substitutes $y = 2$ and $y = 0$ into their integrated expression and subtracts.
	$\frac{512\pi}{315}$ $[\approx 5.11]$	**A1**	oe, or awrt 5.11.
	$\left(\frac{3}{2}\right)^3 \times \text{“}\frac{512\pi}{315}\text{”}$	**M1**	Multiplies their model volume by the cube of 3 over their model height, or integrates with lengths scaled by $\frac{3}{2}$.
	$\frac{192\pi}{35} \approx 17.2\,\text{cm}^3$	**A1**	oe, or awrt 17.2.
		[7]	
Total for Question 14		**11**	

Question	Scheme	Marks	Notes
15(a)	$\cos 5\theta = \cos(4\theta + \theta) = \cos 4\theta \cos\theta - \sin 4\theta \sin\theta$ $\cos 3\theta = \cos(4\theta - \theta) = \cos 4\theta \cos\theta + \sin 4\theta \sin\theta$	**M1 A1**	**M1** Expands $\cos(4\theta + \theta)$ and $\cos(4\theta - \theta)$ by compound-angle formulae. Condone sign slips. **A1** Both expansions correct.
	$\cos 5\theta + \cos 3\theta = 2\cos 4\theta \cos\theta$ $\Rightarrow \cos 4\theta \cos\theta \equiv \frac{1}{2}(\cos 5\theta + \cos 3\theta)$	**A1 AG**	Correct expansions combined without error to the printed identity, with LHS = RHS or a conclusion in words.
		[3]	
15(b)	$V = \pi \int x^2 \frac{\mathrm{d}y}{\mathrm{d}\theta}\,\mathrm{d}\theta,\ \frac{\mathrm{d}y}{\mathrm{d}\theta} = 24\cos\theta$	**M1**	Uses $\pi \int x^2\,\mathrm{d}y$ with x^2 in terms of θ and $\mathrm{d}y$ as their $k\cos\theta\,\mathrm{d}\theta$. Limits not needed.
	$V = 5400\pi \int_0^{\pi/2} \sin^2 2\theta \cos\theta\,\mathrm{d}\theta$	**A1**	Correct integral with limits 0 and $\frac{\pi}{2}$, oe, need not be simplified.
	$2700\pi \int_0^{\pi/2} \left(\cos\theta - \frac{1}{2}\cos 5\theta - \frac{1}{2}\cos 3\theta\right)\mathrm{d}\theta$	**M1**	Simplifies the integrand, by $\sin^2 2\theta = \frac{1}{2}(1 \pm \cos 4\theta)$ and part (a) or otherwise, and integrates at least one term correctly. Condone sign slips.
	$= 2700\pi \left[\sin\theta - \frac{1}{10}\sin 5\theta - \frac{1}{6}\sin 3\theta\right]_0^{\pi/2}$	**A1**	Correct integral, oe, including $5400\pi \left[\frac{4}{3}u^3 - \frac{4}{5}u^5\right]_0^1$ with $u = \sin\theta$.
	$= 2700\pi \left[\left(1 - \frac{1}{10} + \frac{1}{6}\right) - 0\right]$	**dM1**	Substitutes both limits, or the matching limits in a new variable, into their integral and subtracts.
	$V = 2880\pi\ (\mathrm{mm}^3)$	**A1**	Or awrt 9050.
		[6]	
	Total for Question 15	**9**	