CORE PURE / FURTHER CALCULUS

Maclaurin Series

Mark scheme 17 questions · 128 marks

EXAM BOARDS Edexcel 9FM0 · AQA 7367 · OCR A H245 · OCR MEI H645 · CIE 9231

Questions

Question	Marks by part	Marks	Page
Question 1	(a)(i) 2 (a)(ii) 2 (b) 2	6	4
Question 2	(a) 2 (b) 1 (c) 3	6	5
Question 3	(i) 5 (ii) 3	8	6
Question 4	(i) 2 (ii) 2 (iii) 2	6	7
Question 5	(a) 2 (b) 2 (c) 3	7	8
Question 6	(a) 2 (b) 1 (c) 3	6	9
Question 7	(a) 2 (b) 2 (c) 2	6	10
Question 8	(a) 5 (b) 3 (c) 1 (d) 1	10	11
Question 9		5	12
Question 10	(i) 4 (ii) 4	8	13
Question 11		5	14
Question 12	(a) 3 (b) 3	6	15
Question 13	(i) 3 (ii) 4 (iii) 5	12	16
Question 14	(a)(i) 4 (a)(ii) 2 (b) 4	10	17
Question 15	(a) 4 (b) 4	8	18
Question 16	(a) 4 (b) 5	9	19
Question 17	(a)(i) 4 (a)(ii) 2 (b) 4	10	20
	Total	**128**	

Reading the mark scheme

These rules apply throughout, unless a note says otherwise.

Marks

Every mark is worth one, so **M1 A1** on a row is two marks. Only marks coded ft follow through.

Code	Meaning
M1	Method: a correct method, applied to this question. A slip doesn't cost it if the method is clear. A formula must be used, not just quoted.
dM1	Dependent method: as M1, but only if the method mark before it was earned, unless its note names another.
A1	Accuracy: a correct answer or step. It needs the method mark it follows.
A1ft	As A1, following through from an earlier answer that the note names.
A1 AG	The last mark of a part whose answer is printed in the question: complete working, with no errors.
B1	A correct value, fact or statement. It needs no method mark.
B1ft	As B1, following through from an earlier answer that the note names.
E1	A written explanation, interpretation or conclusion. Concluding a proof by induction needs every earlier mark.

Abbreviations

ft and AG qualify a code in the Marks column. The others appear in the notes. A grey line gives the rule where the name alone doesn't.

ft	follow through Correct work from the student's own earlier wrong value earns the mark.	**AG**	answer given The answer is printed in the question, so every step to it must be shown.
cao	correct answer only No follow-through from an earlier error.	**cso**	correct solution only No error anywhere in the part, not just in its answer.
oe	or equivalent	**awrt**	answers which round to
soi	seen or implied Earned if written, or if later working shows it was used.	**isw**	ignore subsequent working A correct answer, once reached, isn't spoiled by later working.
SC	special case Credit, as the note says, for work outside the main method.	**their**	the student's own earlier value
condone	a slip that doesn't cost the mark	[ ]	working the mark doesn't need
s.f.	significant figures	**d.p.**	decimal places

Reading the scheme

- Each row is a line of working, with the marks it earns beside it. Where a row carries two marks, each note starts with its code.
- [***n***] ends each part with its total.
- A shaded *Alternative* is a second method for the same total. Mark an answer by the method it follows.
- Any other valid method earns the equivalent marks, unless the question names a method or says *hence*.
- A sketch earns one B1 for each feature named below its figure.

Accuracy and answers

- Unless a question says otherwise, give answers to 3 significant figures, and angles in degrees to 1 decimal place. Angles in radians take 3 significant figures.
- A final answer with more figures is accepted if it rounds to the scheme's value; one with fewer is not. An answer spoiled by rounding too early loses its accuracy mark.
- Where a question asks for an exact answer, a decimal loses the final accuracy mark. Otherwise either is accepted.
- Wrong or missing units cost nothing unless the question asks for them.
- An omitted $\mathrm{d}\theta$, $\mathrm{d}x$ or $\mathrm{d}u$ in an integral is not penalised where the variable of integration is clear from the integrand and the limits. It is penalised where it is genuinely ambiguous: a substitution part-way through, two variables in play, or limits in the wrong variable.
- A correct answer with no working earns full marks, unless the question needs the working: show that, prove, hence, a method it names, or an exact answer. A wrong answer with no working earns nothing. Calculator answers are treated the same way.
- A correct answer, once reached, isn't spoiled by later working (isw).

Value of g

- Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question states a value, which is then the only one accepted. Answers that use $g = 9.8$ may be given to 2 or 3 significant figures.
- Where a question doesn't state g, an answer worked with $g = 10$ throughout earns the same marks, keeping any value printed in the question. The worked solutions list the final answers with $g = 10$.

Question	Scheme	Marks	Notes
1(a)(i)	$f'(x) = -2e^{-2x}$	**M1**	Differentiates to the form ke^{-2x}, any nonzero constant k.
	$f''(x) = 4e^{-2x}$	**A1**	Both derivatives correct.
		[2]	
1(a)(ii)	$f(0) = 1,\ f'(0) = \text{“}-2\text{”},\ f''(0) = \text{“}4\text{”}$ $e^{-2x} = 1 + x(\text{“}-2\text{”}) + \frac{x^2}{2!}(\text{“}4\text{”}) + \cdots$	**M1**	Uses $f(0) + xf'(0) + \frac{x^2}{2!}f''(0)$ with their derivatives from (a)(i) at $x = 0$. Not for replacing x by $-2x$ in the series for e^x.
	$e^{-2x} = 1 - 2x + 2x^2 + \cdots$	**A1**	
		[2]	
1(b)	$x = \frac{1}{8}$ $e^{-1/4} \approx 1 - 2\left(\frac{1}{8}\right) + 2\left(\frac{1}{8}\right)^2$	**M1**	Substitutes $x = \frac{1}{8}$ into their three-term series from (a)(ii).
	$= 1 - \frac{1}{4} + \frac{1}{32} = \frac{25}{32}$	**A1 AG**	From the correct series, reaching $\frac{25}{32}$ without error; combined arithmetic accepted. Not for a calculator value of $e^{-1/4}$.
		[2]	
Total for Question 1		**6**	

Question	Scheme	Marks	Notes
2(a)	$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$	**M1**	Uses the standard series, or derivatives at $x = 0$ in the Maclaurin formula, with factorial denominators. Need not be simplified.
	$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$	**A1**	All five terms, with the factorials evaluated.
		[2]	
2(b)	$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}$	**B1ft**	ft their five terms in (a) with x replaced by $-x$, signs simplified.
		[1]	
2(c)	$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$ $= 1 + \frac{x^2}{2} + \frac{x^4}{24} + \cdots$	**M1**	Adds their series for e^x and e^{-x} and halves, reaching at least the terms in x^2 and x^4.
	$\cosh^2\left(\frac{x}{2}\right) = \frac{1}{2}\left(\text{“}1 + \frac{x^2}{2} + \frac{x^4}{24}\text{”} + 1\right)$	**dM1**	Adds 1 to their series for $\cosh x$ and divides by 2.
	$= 1 + \frac{1}{4}x^2 + \frac{1}{48}x^4 + \cdots$	**A1**	Correct series, or $a = 1$, $b = \frac{1}{4}$, $c = \frac{1}{48}$ stated.
		[3]	
	Total for Question 2	**6**	

Question	Scheme	Marks	Notes
3(i)	$\sin 2x = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \cdots$ $\cos 3x = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} + \cdots$	**M1**	Replaces x by $2x$ in the $\sin x$ series and by $3x$ in the $\cos x$ series, at least two terms each. Need not be simplified.
	$\cos x - \cos 3x = \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)$ $- \text{“}\left(1 - \frac{9x^2}{2} + \frac{81x^4}{24}\right)\text{”}$ $= 4x^2 - \frac{10}{3}x^4 + \cdots$	**M1**	Subtracts their $\cos 3x$ series from the $\cos x$ series, reaching terms in x^2 and x^4. Condone sign slips.
	$\text{“}\left(2x - \frac{4}{3}x^3 + \frac{4}{15}x^5\right)\text{”} \text{“}\left(4x^2 - \frac{10}{3}x^4\right)\text{”}$ $= 8x^3 - \frac{20}{3}x^5 - \frac{16}{3}x^5 + \cdots$	**dM1**	Multiplies their two series, forming the x^3 term and both products giving x^5. Condone slips in coefficients.
	$\sin 2x(\cos x - \cos 3x) = 8x^3 - 12x^5 + \cdots$	**A1 A1**	**A1** $8x^3$ **A1** $-12x^5$
		[5]	
3(ii)	$2\left(\text{“}8x^3 - 12x^5\text{”}\right) = x$	**M1**	Replaces the LHS by twice their series from (i), of the form $ax^3 + bx^5$, and equates to x.
	$24x^4 - 16x^2 + 1 = 0$ $\Rightarrow x^2 = \frac{16 \pm \sqrt{256-96}}{48} = \ldots$	**dM1**	Divides by x and solves their three-term quadratic in x^2 by the formula with their values or completing the square.
	$x^2 = \frac{4-\sqrt{10}}{12} \Rightarrow x = \sqrt{\frac{4-\sqrt{10}}{12}}$	**A1 AG**	From a correct quadratic in x^2 without error, selecting the smaller value of x^2 and the positive root.
		[3]	
	Total for Question 3	**8**	

Question	Scheme	Marks	Notes
4(i)	$f'(x) = \frac{\cos x}{1+\sin x}$ $f''(x) = \frac{(-\sin x)(1+\sin x)-\cos x \cos x}{(1+\sin x)^2}$	**M1**	Obtains $f'(x)$ of the form $\frac{k\cos x}{1+\sin x}$ and differentiates it by the quotient or product rule. Condone sign slips.
	$f''(x) = -\frac{1}{1+\sin x}$	**A1**	Any correct form, simplified or not.
		[2]	
4(ii)	$f(0) = 0,\ f'(0) = 1,\ f''(0) = \text{“}-1\text{”}$ $f(x) = 0 + x(1) + \frac{x^2}{2!}(\text{“}-1\text{”}) + \cdots$	**M1**	Substitutes $x = 0$ into their f' and f'' and uses $f(0) + xf'(0) + \frac{x^2}{2!}f''(0)$, or composes the series for $\ln(1+u)$ and $\sin x$.
	$f(x) = x - \frac{x^2}{2} + \cdots$	**A1**	
		[2]	
4(iii)	$\int_0^{2/5} \text{“}x - \frac{x^2}{2}\text{”}\, \mathrm{d}x = \left[\frac{x^2}{2} - \frac{x^3}{6}\right]_0^{2/5}$ $= \frac{(2/5)^2}{2} - \frac{(2/5)^3}{6}$	**M1**	Integrates their two-term series, raising the power of at least one term, and substitutes $x = \frac{2}{5}$.
	$= \frac{26}{375} \approx 0.069333$	**A1**	awrt 0.069333; $\frac{26}{375}$ alone or fewer than 6 d.p. scores A0.
		[2]	
	Total for Question 4	**6**	

Question	Scheme	Marks	Notes
5(a)	$\ln\left(\frac{1+2x}{1-x}\right) = \ln(1+2x) - \ln(1-x)$ $\ln(1+2x) = 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} + \cdots$ $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$	**M1**	Splits the logarithm and substitutes $2x$ and $-x$ into the $\ln(1+x)$ series, to at least the x^3 terms. Condone sign slips.
	$\left(2x - 2x^2 + \frac{8}{3}x^3\right) - \left(-x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right)$ $= 3x - \frac{3}{2}x^2 + 3x^3$	**A1 AG**	Correct expansions of both logarithms combined without error to the printed terms.
		[2]	
5(b)	Series valid only for $-\frac{1}{2} < x \leq \frac{1}{2}$	**B1**	Accept $\lvert 2x\rvert < 1$ or $\lvert x\rvert < \frac{1}{2}$; condone the endpoints.
	$x = \frac{5}{8} > \frac{1}{2}$, outside the range of validity	**E1**	States that $\frac{5}{8}$ lies outside a correct range, so the series is not valid there. Not for saying only that the answer is inaccurate.
		[2]	
5(c)	$\ln 2 \approx 3\left(\frac{1}{4}\right) - \frac{3}{2}\left(\frac{1}{4}\right)^2 + 3\left(\frac{1}{4}\right)^3$ $\ln 8 = 3\ln 2 \approx 3 \times \ldots$	**M1**	Substitutes $x = \frac{1}{4}$ into the three terms as an approximation for $\ln 2$ and multiplies by 3. Condone arithmetic slips.
	$\ln 8 \approx \frac{135}{64} \approx 2.11$	**A1**	$\frac{135}{64}$ oe, or awrt 2.11.
	$x = \frac{1}{4} \Rightarrow \frac{1+2x}{1-x} = \frac{3/2}{3/4} = 2$ $-\frac{1}{2} < \frac{1}{4} \leq \frac{1}{2}$, within the range of validity	**E1**	Shows $\frac{1+2x}{1-x} = 2$ at $x = \frac{1}{4}$ and states that $\frac{1}{4}$ lies in the range of validity.
		[3]	
	Total for Question 5	**7**	

Question	Scheme	Marks	Notes
6(a)	$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$	**M1**	Writes the series for $\ln(1-x)$ or $-\ln(1-x)$ with at least two terms and an indication it continues. Condone a sign slip.
	$0 < x < 1 \Rightarrow \frac{x^2}{2} + \frac{x^3}{3} + \cdots > 0$ $\Rightarrow -\ln(1-x) > x$	**A1 AG**	Correct series, with the terms after x stated positive for $0 < x < 1$, leading to the printed inequality. A0 without the reason.
		[2]	
6(b)	$x = 1 - \frac{1}{t}$, $t > 1 \Rightarrow 0 < x < 1$ $-\ln\left(\frac{1}{t}\right) > 1 - \frac{1}{t} \Rightarrow \ln t > 1 - \frac{1}{t}$	**B1**	Substitutes $x = 1 - \frac{1}{t}$ into (a), states $0 < x < 1$ for $t > 1$, and reaches the printed result without error.
		[1]	
6(c)	$\ln\left(\frac{5}{4}\right) > 1 - \frac{1}{5/4}$	**M1**	Substitutes $t = \frac{5}{4}$ into the inequality from (b). Comparing calculator values scores nothing in this part.
	$\ln\left(\frac{5}{4}\right) > \frac{1}{5}$	**A1**	
	$5\ln\left(\frac{5}{4}\right) > 1 = \ln e$ $\Rightarrow \left(\frac{5}{4}\right)^5 > e$	**A1**	cso. Multiplies by 5 and compares with $\ln e$, or exponentiates, concluding $\left(\frac{5}{4}\right)^5$ is greater.
		[3]	
Total for Question 6		**6**	

Question	Scheme	Marks	Notes
7(a)	$\ln(1+2x)-\ln(1-x)$ $=\left(2x-\frac{(2x)^2}{2}+\frac{(2x)^3}{3}\right)$ $-\left(-x-\frac{x^2}{2}-\frac{x^3}{3}\right)$	**M1**	Splits the logarithm as a difference and expands both with $\ln(1+u)$, $u=2x$ and $u=-x$, at least to x^2. Condone sign slips.
	$3x-\frac{3}{2}x^2+3x^3$	**A1**	oe; isw.
		[2]	
7(b)	$\frac{1+2x}{1-x}=\frac{10}{7}\Rightarrow x=\frac{1}{8}$ $\ln\left(\frac{10}{7}\right)\approx 3\left(\frac{1}{8}\right)-\frac{3}{2}\left(\frac{1}{8}\right)^2$ $+3\left(\frac{1}{8}\right)^3$	**M1**	Solves $\frac{1+2x}{1-x}=\frac{10}{7}$ for x and substitutes their x into their three terms from (a).
	$=\frac{192}{512}-\frac{12}{512}+\frac{3}{512}=\frac{183}{512}$	**A1 AG**	Correct substitution of $x=\frac{1}{8}$ into correct terms, reaching $\frac{183}{512}$ without error. Not for decimal comparison alone.
		[2]	
7(c)	$\frac{1+2x}{1-x}=10\Rightarrow x=\frac{3}{4}$	**B1**	
	$\ln(1+2x)$ series valid only for $-\frac{1}{2}<x\leq\frac{1}{2}$ $x=\frac{3}{4}$ is outside this range	**E1**	States $x=\frac{3}{4}$ lies outside the range of validity, with $\lvert x\rvert<\frac{1}{2}$, $x\leq\frac{1}{2}$ or $-\frac{1}{2}<x\leq\frac{1}{2}$ seen. Not for "x too large" alone.
		[2]	
	Total for Question 7	**6**	

Question	Scheme	Marks	Notes
8(a)	$\sin\left(\frac{x}{2}\right) = \frac{x}{2} - \frac{x^3}{48} + \frac{x^5}{3840} + \cdots$ $\cos\left(\frac{x}{3}\right) = 1 - \frac{x^2}{18} + \frac{x^4}{1944} + \cdots$	**M1**	Replaces x by $\frac{x}{2}$ in the sin series and by $\frac{x}{3}$ in the cos series, at least two terms each. Need not be simplified.
	$\frac{\sin(x/2)}{x/2} = 1 - \frac{x^2}{24} + \frac{x^4}{1920} + \cdots$	**M1**	Divides each term of their series for $\sin\left(\frac{x}{2}\right)$ by $\frac{x}{2}$. Condone one arithmetic slip.
	$\left(1 - \frac{x^2}{24} + \frac{x^4}{1920}\right)\left(1 - \frac{x^2}{18} + \frac{x^4}{1944}\right)$ $= 1 - \left(\frac{1}{24} + \frac{1}{18}\right)x^2$ $+ \left(\frac{1}{1920} + \frac{1}{1944} + \frac{1}{432}\right)x^4 + \cdots$	**dM1**	Multiplies their two series, gathering the x^2 and x^4 terms. Condone one missing x^4 product and arithmetic slips.
	$= 1 - \frac{7}{72}x^2 + \frac{521}{155520}x^4$	**A1 A1**	**A1** $1 - \frac{7}{72}x^2$, simplified. **A1** $\frac{521}{155520}x^4$ cao; a decimal coefficient scores A0.
		[5]	
8(b)	$\int\left(\frac{1}{x} - \frac{7x}{72} + \frac{521x^3}{155520}\right)\mathrm{d}x$ $= \ln x - \frac{7x^2}{144} + \frac{521x^4}{622080}$	**M1**	Divides their series from (a), of the form $1 + px^2 + qx^4$, by x and integrates, with $\frac{1}{x} \to \ln x$ or another power increased by 1.
	$\left[\ln x - \frac{7x^2}{144} + \frac{521x^4}{622080}\right]_{\pi/6}^{2\pi/3} = \ldots$ $\left[= \ln 4 - \frac{35\pi^2}{1728} + \frac{8857\pi^4}{53747712}\right]$	**dM1**	Substitutes $\frac{2\pi}{3}$ and $\frac{\pi}{6}$ into their integral and subtracts. Implied by a correct answer.
	≈ 1.20244	**A1**	awrt 1.20244.
		[3]	
8(c)	1.20169	**B1**	
		[1]	
8(d)	$1.20244 - 1.20169 = 0.00075$ Overestimate; accurate to 3 d.p. but not to 4 d.p.	**E1**	Makes any correct comment on the accuracy of (b), such as accurate to 3 d.p. (or 2 d.p.), or an error of 0.00075 oe. Direction not required. "Close" alone is not enough.
		[1]	
	Total for Question 8	**10**	

Question	Scheme	Marks	Notes
9	$\frac{x}{\sin x} = \frac{1}{1-\frac{x^2}{6}+\frac{x^4}{120}-\frac{x^6}{5040}+\cdots}$	**M1**	Uses the series for $\sin x$, with at least three terms, divided by x. Condone factorial or sign slips.
	$(1+ax^2+bx^4+cx^6)$ $\times\left(1-\frac{x^2}{6}+\frac{x^4}{120}-\frac{x^6}{5040}\right)=1$	**M1**	Multiplies by their series and sets the product equal to 1, collecting at least the terms in x^2 and x^4.
	$a-\frac{1}{6}=0,\ b-\frac{a}{6}+\frac{1}{120}=0,$ $c-\frac{b}{6}+\frac{a}{120}-\frac{1}{5040}=0$ $\Rightarrow a=\ldots,\ b=\ldots,\ c=\ldots$	**dM1**	Equates their coefficients of x^2, x^4 and x^6 to zero and solves for all three. Condone arithmetic slips.
	$a=\frac{1}{6},\ b=\frac{7}{360}$	**A1**	Both exact.
	$c=\frac{31}{15120}$	**A1**	
	Total for Question 9	**5**	

Question	Scheme	Marks	Notes
10(i)	$\ln\left(\frac{(1+x)^2}{1-2x}\right) = 2\ln(1+x) - \ln(1-2x)$	**M1**	Splits into multiples of $\ln(1+x)$ and $\ln(1-2x)$ by log laws. Condone one sign or coefficient slip.
	$2\ln(1+x) = 2x - x^2 + \frac{2}{3}x^3 - \cdots$ $\ln(1-2x) = -2x - \frac{(2x)^2}{2} - \frac{(2x)^3}{3} - \cdots$	**M1**	Uses $2x$ for x in the series for $\ln(1-x)$, or $-2x$ in that for $\ln(1+x)$, to the x^3 term. Condone sign slips.
	$4x + x^2 + \frac{10}{3}x^3$	**A1**	All three terms correct, oe.
	$-\frac{1}{2} \leq x < \frac{1}{2}$	**B1**	Both endpoints, with the correct strict and non-strict inequalities.
		[4]	
10(ii)	$(1+x)^2 = 2(1-2x)$ $\Rightarrow x^2 + 6x - 1 = 0 \Rightarrow x = \ldots$	**M1**	Clears the fraction to a three-term quadratic and solves it by any method, calculator allowed. Condone sign slips.
	$x = \sqrt{10} - 3$	**A1**	Or awrt 0.162, with $-3 - \sqrt{10}$ rejected or not given.
	$\ln 2 \approx 4(\text{“}\sqrt{10} - 3\text{”}) + (\text{“}\sqrt{10} - 3\text{”})^2$ $+ \frac{10}{3}(\text{“}\sqrt{10} - 3\text{”})^3$	**M1**	Substitutes their root of the quadratic into their three-term series from (i) as the approximation to $\ln 2$.
	$\ln 2 \approx 0.690$	**A1**	0.690, to 3 d.p.
		[4]	
	Total for Question 10	**8**	

Question	Scheme	Marks	Notes
11	$e^{x^2} = 1 + x^2 + \cdots \Rightarrow e^{x^2} - x = 1 - x + x^2 + \cdots$	**M1**	Replaces x by x^2 in the series for e^x, reaching at least $1 + x^2$, and subtracts x.
	$e^{e^{x^2}-x} = e \times e^u$, $u = -x + x^2 + \cdots$	**M1**	Takes out the factor e, writing the expression as $e \times e^u$ with u their exponent less 1.
	$e^u = 1 + \text{“}(-x + x^2)\text{”} + \frac{1}{2}\text{“}(-x + x^2)\text{”}^2 + \cdots$ $= 1 - x + \ldots x^2 + \cdots$	**dM1**	Substitutes their u, of the form $-x + kx^2$, into $1 + u + \frac{u^2}{2}$ and collects terms to x^2. Condone slips in squaring.
	$e^{e^{x^2}-x} = e - ex + \frac{3e}{2}x^2 + \cdots$	**A1 A1**	**A1** $e - ex$, oe such as $e(1 - x + \cdots)$. **A1** $\frac{3e}{2}x^2$ oe.
	Total for Question 11	**5**	

Question	Scheme	Marks	Notes
12(a)	$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$ $= 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + \cdots$ $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$ $= x + \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^7}{5040} + \cdots$	**M1**	Adds and subtracts their series for e^x and e^{-x} and halves, giving at least three terms of each series. Condone factorial slips.
	$x\cosh x - \sinh x = \left(\frac{1}{2} - \frac{1}{6}\right)x^3$ $+ \left(\frac{1}{24} - \frac{1}{120}\right)x^5 + \left(\frac{1}{720} - \frac{1}{5040}\right)x^7 + \cdots$	**dM1**	Multiplies their $\cosh x$ series by x, subtracts their $\sinh x$ series and collects like powers of x.
	$x\cosh x - \sinh x = \frac{x^3}{3} + \frac{x^5}{30} + \frac{x^7}{840} + \cdots$	**A1**	
		[3]	
12(b)	$\frac{(ix)^3}{3} + \frac{(ix)^5}{30} + \frac{(ix)^7}{840}$ $= -\frac{ix^3}{3} + \frac{ix^5}{30} - \frac{ix^7}{840}$	**M1**	Replaces x by ix in their three-term series from (a) and simplifies the powers of i. Condone one sign slip.
	$ix\cosh(ix) - \sinh(ix) = ix\cos x - i\sin x$	**B1**	Both identities used, giving $i(x\cos x - \sin x)$ oe.
	$x\cos x - \sin x = -\frac{x^3}{3} + \frac{x^5}{30} - \frac{x^7}{840} + \cdots$	**A1**	cso.
		[3]	
Total for Question 12		**6**	

Question	Scheme	Marks	Notes
13(i)	$x = \tan y \Rightarrow 1 = \sec^2 y \frac{\mathrm{d}y}{\mathrm{d}x}$	**M1**	Writes $x = \tan y$ and differentiates with respect to x or y, reaching $\sec^2 y$. Condone a sign slip.
	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y}$	**M1**	Uses $\sec^2 y = 1 + \tan^2 y$, or an equivalent identity, with $\tan y = x$ to write the derivative in terms of x.
	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2}$	**A1 AG**	Correct differentiated equation leading without error to the printed result.
		[3]	
13(ii)	$\frac{1}{1+x^2} = 1 + (-x^2) + (-x^2)^2 + \cdots$	**M1**	Expands $(1 + x^2)^{-1}$ by the binomial or geometric series to at least three terms. Condone sign slips.
	$= 1 - x^2 + x^4 - \cdots$	**A1**	
	$\arctan x = \int$ "$1 - x^2 + x^4$" $\mathrm{d}x$ $\arctan 0 = 0 \Rightarrow c = \ldots$	**dM1**	Integrates their series term by term, raising each power by 1, and finds the constant from $\arctan 0 = 0$.
	$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots$	**A1**	
		[4]	
13(iii)	$\ln(1 + t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots$ with $t =$ "$x - \frac{x^3}{3}$"	**M1**	Uses the $\ln(1 + t)$ series, with at least three terms, and t replaced by their series for $\arctan x$.
	$t^2 = x^2 - \frac{2}{3}x^4 + \cdots$, $t^3 = x^3 + \cdots$, $t^4 = x^4 + \cdots$	**M1**	Expands at least two of t^2, t^3, t^4 from their series, keeping terms up to x^4. Condone coefficient slips.
	$x - \frac{x^3}{3} - \frac{1}{2}\left(x^2 - \frac{2}{3}x^4\right) + \frac{x^3}{3} - \frac{x^4}{4}$	**dM1**	Substitutes their powers into the series and collects the terms in each power up to x^4.
	$\ln(1 + \arctan x) = x - \frac{x^2}{2} + \frac{x^4}{12} + \cdots$	**A1 A1**	**A1** Any two of x, $-\frac{x^2}{2}$, $\frac{x^4}{12}$ correct. **A1** All correct, with no x^3 term other than $0x^3$. cso.
		[5]	
	Total for Question 13	**12**	

Question	Scheme	Marks	Notes
14(a)(i)	$f'(x) = 2\operatorname{sech} x\,(-\operatorname{sech} x \tanh x)$ $\quad = -2\operatorname{sech}^2 x \tanh x$	**M1 A1**	**M1** Uses the chain rule on $(\operatorname{sech} x)^2$, or a quotient rule on $\frac{1}{\cosh^2 x}$, to obtain the form $k\operatorname{sech}^2 x \tanh x$. **A1** oe, such as $-\frac{2\sinh x}{\cosh^3 x}$.
	$f'' = -2f'\tanh x - 2f\operatorname{sech}^2 x = 4f - 6f^2$ $f''' = 4f' - 12ff'$ $f^{(4)} = 4f'' - 12\big((f')^2 + ff''\big)$ $\Rightarrow f^{(4)}(0) = \ldots$	**dM1**	Differentiates three more times, in x or in terms of f, and substitutes $x = 0$. Condone sign and coefficient slips.
	$f(0) = 1,\ f'(0) = 0,\ f''(0) = -2$ $f^{(4)}(0) = 4(-2) - 12\big(0 + 1(-2)\big)$ $\quad = 16$	**A1 AG**	Correct derivatives with correct values at 0 substituted, leading without error to 16.
		[4]	
14(a)(ii)	$f(x) = 1 + \frac{x^2}{2!}(-2) + \frac{x^4}{4!}(16) + \cdots$	**M1**	Substitutes their $f(0)$, $f''(0)$ from (a)(i) and 16 into the Maclaurin formula with factorial denominators.
	$\operatorname{sech}^2 x = 1 - x^2 + \frac{2}{3}x^4 + \cdots$	**A1**	
		[2]	
14(b)	$\cos(\sqrt{2}\,x) = 1 - \frac{(\sqrt{2}\,x)^2}{2!} + \frac{(\sqrt{2}\,x)^4}{4!} + \cdots$	**M1**	Substitutes $\sqrt{2}\,x$ for x in the series for $\cos x$, reaching the x^4 term.
	$\cos(\sqrt{2}\,x) = 1 - x^2 + \frac{1}{6}x^4 + \cdots$	**A1**	
	$\frac{\operatorname{sech}^2 x - \cos(\sqrt{2}\,x)}{x^4}$ $= \frac{\left(\frac{2}{3} - \text{“}\frac{1}{6}\text{”}\right)x^4 + \cdots}{x^4}$	**M1**	Subtracts their series for $\cos(\sqrt{2}\,x)$ from $1 - x^2 + \frac{2}{3}x^4$ and divides by x^4. Condone slips in the coefficients.
	$= \frac{1}{2} +$ terms in positive powers of x $\to \frac{1}{2}$ as $x \to 0$	**A1 AG**	Correct series leading without error to $\frac{1}{2}$, with a statement that the terms in positive powers of x tend to 0.
		[4]	
	Total for Question 14	**10**	

Question	Scheme	Marks	Notes
15(a)	$2(1-\cos x) = 2\sum_{r=1}^{\infty}(-1)^{r+1}\frac{x^{2r}}{(2r)!}$ $x\sin x = \sum_{r=0}^{\infty}(-1)^{r}\frac{x^{2r+2}}{(2r+1)!}$	**M1**	Uses the general terms of the booklet's $\cos x$ and $\sin x$ series to write general terms for both expressions. Condone sign and index slips.
	$x\sin x = \sum_{r=1}^{\infty}(-1)^{r-1}\frac{x^{2r}}{(2r-1)!}$	**A1**	Both general terms correct in powers of x^{2r}, oe. Summation limits need not be stated.
	$\frac{2(-1)^{r+1}}{(2r)!} - \frac{(-1)^{r-1}}{(2r-1)!}$ $= (-1)^{r}\frac{-2+2r}{(2r)!}$	**M1**	Subtracts their coefficients of x^{2r} and writes them over $(2r)!$, using $(2r)! = 2r(2r-1)!$. Condone sign slips.
	$\Rightarrow (-1)^{r}\frac{2(r-1)}{(2r)!}x^{2r}$	**A1 AG**	Correct difference of coefficients leading without error to the printed general term, including x^{2r}; combined steps accepted.
		[4]	
15(b)	$2(1-\cos x) - x\sin x = \frac{2(2-1)}{4!}x^4 + \cdots$	**M1**	Finds the first nonzero term of the numerator, by $r=2$ in (a) or by expanding the series.
	$= \frac{x^4}{12} + \cdots$	**A1**	
	$(1-\cos x)^2 = \left(\frac{x^2}{2} - \frac{x^4}{24} + \cdots\right)^2$ $= \frac{x^4}{4} + \cdots$ $\frac{2(1-\cos x)-x\sin x}{(1-\cos x)^2} = \frac{\text{“}\frac{x^4}{12}\text{”}+\cdots}{\frac{x^4}{4}+\cdots}$	**M1**	Squares their series for $1-\cos x$ to a leading term in x^4 and divides the leading terms. Condone coefficient slips.
	$\lim_{x\to 0}[\ldots] = \frac{1/12}{1/4} = \frac{1}{3}$	**A1 AG**	Correct quotient of both series leading without error to $\frac{1}{3}$, with the limit taken.
		[4]	
	Total for Question 15	**8**	

Question	Scheme	Marks	Notes
16(a)	$\cosh y = \frac{x+3}{x+2}$ $\Rightarrow \sinh y \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x+2)-(x+3)}{(x+2)^2}$ $= -\frac{1}{(x+2)^2}$	**M1 A1**	**M1** Differentiates $\cosh y$ implicitly and the right side by the quotient or product rule, or differentiates arcosh by the chain rule. Condone sign slips. **A1** Correct equation, oe; need not be simplified.
	$\sinh y = \sqrt{\left(\frac{x+3}{x+2}\right)^2 - 1} = \frac{\sqrt{2x+5}}{x+2}$	**M1**	Uses $\sinh^2 y = \cosh^2 y - 1$ with their $\cosh y$ to write $\sinh y$ in terms of x. Condone algebraic slips.
	$\frac{\sqrt{2x+5}}{x+2} \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{(x+2)^2}$ $\Rightarrow (x+2)\sqrt{5+2x} \frac{\mathrm{d}y}{\mathrm{d}x} + 1 = 0$	**A1 AG**	Correct substituted equation leading without error to the printed result. Condone the positive root taken without comment.
		[4]	
16(b)	$(x+2)\frac{\mathrm{d}y}{\mathrm{d}x} + (5+2x)^{-\frac{1}{2}} = 0$ $\Rightarrow (x+2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - (5+2x)^{-\frac{3}{2}} = 0$	**M1 A1**	**M1** Differentiates the relation from (a), in any arrangement, or their $\frac{\mathrm{d}y}{\mathrm{d}x}$ from it, using the product rule to obtain $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. Condone sign and coefficient slips. **A1** Any correct relation or expression for $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$; need not be simplified.
	$y(0) = \operatorname{arcosh} \frac{3}{2} = \ln\left(\frac{3}{2} + \sqrt{\frac{9}{4} - 1}\right)$ $= \ln\left(\frac{3+\sqrt{5}}{2}\right)$	**B1**	Needs the logarithmic form; arcosh $\frac{3}{2}$ alone is not enough.
	$y'(0) = -\frac{1}{2\sqrt{5}}$ $2y''(0) - \frac{1}{2\sqrt{5}} - \frac{1}{5\sqrt{5}} = 0 \Rightarrow y''(0) = \ldots$ $y = y(0) + xy'(0) + \frac{x^2}{2!}y''(0)$	**M1**	Substitutes $x = 0$ into their expressions for $y'(0)$ and $y''(0)$ and uses them in the Maclaurin formula, including the $\frac{1}{2!}$.
	$a = 1,\ b = -\frac{1}{2\sqrt{5}},\ c = \frac{7}{40\sqrt{5}}$	**A1**	All three, oe such as $-\frac{\sqrt{5}}{10}$ and $\frac{7\sqrt{5}}{200}$, or awrt -0.224 and 0.0783; or the series in the given form.
		[5]	
	Total for Question 16	**9**	

Question	Scheme	Marks	Notes
17(a)(i)	$\frac{\mathrm{d}y}{\mathrm{d}x} = n(1+\cosh x)^{n-1}\sinh x$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = n(n-1)(1+\cosh x)^{n-2}\sinh^2 x$ $\quad + n(1+\cosh x)^{n-1}\cosh x$	**M1 A1**	**M1** Differentiates twice, the second time by the product rule, reaching terms in $(1+\cosh x)^{n-2}\sinh^2 x$ and $(1+\cosh x)^{n-1}\cosh x$. Condone coefficient slips. **A1** Correct, unsimplified.
	$\sinh^2 x = (\cosh x - 1)(\cosh x + 1)$ $\Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = n(1+\cosh x)^{n-1}$ $\quad \times ((n-1)(\cosh x - 1) + \cosh x)$	**M1**	Replaces $\sinh^2 x$ by $\cosh^2 x - 1$ oe and takes out the factor $(1+\cosh x)^{n-1}$. Condone sign slips.
	$= n(1+\cosh x)^{n-1}\left(n(1+\cosh x) - (2n-1)\right)$ $= n^2(1+\cosh x)^n - n(2n-1)(1+\cosh x)^{n-1}$	**A1 AG**	Printed result reached without error from a correct factorised form; combined steps accepted.
		[4]	
17(a)(ii)	$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(1+\cosh x)^{n-1} = (n-1)^2(1+\cosh x)^{n-1}$ $\quad -(n-1)(2n-3)(1+\cosh x)^{n-2}$	**M1**	Applies (a)(i) to both terms, replacing n by $n-1$ for the second, or differentiates twice by another valid method. Condone coefficient slips.
	$\frac{\mathrm{d}^4y}{\mathrm{d}x^4} = n^4(1+\cosh x)^n$ $\quad - n(2n-1)(2n^2-2n+1)(1+\cosh x)^{n-1}$ $\quad + n(n-1)(2n-1)(2n-3)(1+\cosh x)^{n-2}$	**A1**	Any correct equivalent; need not be collected.
		[2]	
17(b)	$y(0) = 2^n$ $y''(0) = n^2 2^n - n(2n-1)2^{n-1} = \ldots$ $y^{(4)}(0) = n^4 2^n - n(2n-1)(2n^2-2n+1)2^{n-1}$ $\quad + n(n-1)(2n-1)(2n-3)2^{n-2} = \ldots$	**M1**	Substitutes $\cosh 0 = 1$ into y and their (a)(i) and (a)(ii) expressions in powers of $(1+\cosh x)$.
	$y''(0) = n2^{n-1}$, $y^{(4)}(0) = n(3n-1)2^{n-2}$	**A1**	Both correct, oe.
	$y = 2^n + \frac{x^2}{2!}$"$n2^{n-1}$" $+ \frac{x^4}{4!}$"$n(3n-1)2^{n-2}$" $+ \cdots$	**dM1**	Uses the Maclaurin formula with their values and 2!, 4!, with no odd-power terms. No justification that the odd derivatives vanish is needed.
	$y = 2^n + n2^{n-2}x^2 + \frac{n(3n-1)2^{n-5}}{3}x^4 + \cdots$	**A1**	All three terms, with x^2 coefficient $n2^{n-2}$ or $\frac{n2^n}{4}$, and x^4 coefficient $\frac{n(3n-1)2^{n-5}}{3}$ or $\frac{n(3n-1)2^n}{96}$ only. A0 for any other form, such as $\frac{n(3n-1)2^{n-2}}{24}$.
		[4]	
	Total for Question 17	**10**	