CORE PURE / FURTHER CALCULUS

Arc Length

Mark scheme 12 questions · 94 marks

EXAM BOARDS AQA 7367 · CIE 9231 · Edexcel FP2 9FM0/4A · OCR A Y545

Questions

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	Total	**94**	

Reading the mark scheme

These rules apply throughout, unless a note says otherwise.

Marks

Every mark is worth one, so **M1 A1** on a row is two marks. Only marks coded ft follow through.

Code	Meaning
M1	Method: a correct method, applied to this question. A slip doesn't cost it if the method is clear. A formula must be used, not just quoted.
dM1	Dependent method: as M1, but only if the method mark before it was earned, unless its note names another.
A1	Accuracy: a correct answer or step. It needs the method mark it follows.
A1ft	As A1, following through from an earlier answer that the note names.
A1 AG	The last accuracy mark of the working that reaches a printed result: complete working, with no errors.
B1	A correct value, fact or statement. It needs no method mark.
B1ft	As B1, following through from an earlier answer that the note names.
E1	A written explanation, interpretation or conclusion. Concluding a proof by induction needs every earlier mark.

Abbreviations

ft and AG qualify a code in the Marks column. The others appear in the notes. A grey line gives the rule where the name alone doesn't.

ft	follow through Correct work from the student's own earlier wrong value earns the mark.	**AG**	answer given The answer is printed in the question, so every step to it must be shown.
cao	correct answer only No follow-through from an earlier error.	**cso**	correct solution only No error anywhere in the part, not just in its answer.
oe	or equivalent	**awrt**	answers which round to
soi	seen or implied Earned if written, or if later working shows it was used.	**isw**	ignore subsequent working A correct answer, once reached, isn't spoiled by later working.
SC	special case Credit, as the note says, for work outside the main method.	**their**	the student's own earlier value
condone	a slip that doesn't cost the mark	[ ]	working the mark doesn't need
s.f.	significant figures	**d.p.**	decimal places

Reading the scheme

- Each row is a line of working, with the marks it earns beside it. Where a row carries two marks, each note starts with its code.
- [$\boldsymbol{n}$] ends each part with its total.
- A shaded *Alternative* is a second method for the same total. Mark an answer by the method it follows.
- Any other valid method earns the equivalent marks, unless the question names a method or says *hence*.
- A sketch earns one B1 for each feature named below its figure.

Accuracy and answers

- Unless a question says otherwise, give answers to 3 significant figures, and angles in degrees to 1 decimal place. Angles in radians take 3 significant figures.
- A final answer with more figures is accepted if it rounds to the scheme's value; one with fewer is not. An answer spoiled by rounding too early loses its accuracy mark.
- Where a question asks for an exact answer, a decimal loses the final accuracy mark. Otherwise either is accepted.
- Wrong or missing units cost nothing unless the question asks for them.
- An omitted $\mathrm{d}\theta$, $\mathrm{d}x$ or $\mathrm{d}u$ in an integral is not penalised where the variable of integration is clear from the integrand and the limits. It is penalised where it is genuinely ambiguous: a substitution part-way through, two variables in play, or limits in the wrong variable.
- A correct answer with no working earns full marks, unless the question needs the working: show that, prove, hence, a method it names, or an exact answer. A wrong answer with no working earns nothing. Calculator answers are treated the same way.
- A correct answer, once reached, isn't spoiled by later working (isw).

Value of g

- Take $g = 9.8\,\mathrm{m\,s^{-2}}$ unless a question states a value, which is then the only one accepted. Answers that use $g = 9.8$ may be given to 2 or 3 significant figures.
- Where a question doesn't state g, an answer worked with $g = 10$ throughout earns the same marks, keeping any value printed in the question. The worked solutions list the final answers with $g = 10$.

Question	Scheme	Marks	Notes
1	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2} \times \frac{2\cos 2x}{\sin 2x} = \cot 2x$	**M1 A1**	**M1** Differentiates to the form $k\cot 2x$ or $\frac{k\cos 2x}{\sin 2x}$. **A1** Any correct form, such as $\frac{\cos 2x}{\sin 2x}$.
	$s = \int \sqrt{1+\cot^2 2x}\,\mathrm{d}x$ $1+\cot^2 2x = \operatorname{cosec}^2 2x$	**M1**	Uses $1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$ with their $k\cot 2x$ and simplifies by a correct identity to a single term, such as $\operatorname{cosec}^2 2x$.
	$s = \int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \operatorname{cosec} 2x\,\mathrm{d}x$	**A1**	Correct integrand, oe such as $\frac{1}{\sin 2x}$.
	$s = \frac{1}{2}\left[\ln(\tan x)\right]_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}$ $= \frac{1}{2}\left(\ln\sqrt{3} - \ln\frac{1}{\sqrt{3}}\right)$	**dM1**	Integrates their $\operatorname{cosec} 2x$ to a log form, such as $k\ln(\tan x)$ or $k\ln\lvert\operatorname{cosec} 2x + \cot 2x\rvert$, and substitutes both limits. Condone coefficient slips.
	$s = \frac{1}{2}\ln 3$	**A1**	cso. Any $a\ln b$ with a rational and b an integer, such as $\frac{1}{4}\ln 9$. Not $\ln\sqrt{3}$ alone.
	Total for Question 1	**6**	

Question	Scheme	Marks	Notes
2	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\sqrt{x^2-1} + \frac{x^2}{2\sqrt{x^2-1}}$ $- \frac{1}{2\sqrt{x^2-1}}$	**M1**	Uses the product rule on the first term, with two terms, and $\frac{1}{\sqrt{x^2-1}}$ for $\operatorname{arcosh} x$. Condone sign and coefficient slips.
	$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{x^2-1}$	**A1**	Simplified to $\sqrt{x^2-1}$ oe single term.
	$s = \int_2^5 \sqrt{1 + \text{“}(x^2-1)\text{”}}\,\mathrm{d}x$	**M1**	Uses the arc length formula with their $\frac{\mathrm{d}y}{\mathrm{d}x}$, squaring it and adding 1 to give a single expression. Limits need not be seen.
	$s = \int_2^5 x\,\mathrm{d}x$	**A1**	Integrand x; $\sqrt{x^2}$ alone is not enough.
	$\left[\frac{x^2}{2}\right]_2^5 = \frac{25}{2} - 2$	**dM1**	Integrates their integrand, at least one power increased by 1, and substitutes 5 and 2, subtracting.
	$s = \frac{21}{2}$	**A1**	$\frac{21}{2}$ or 10.5.
	Total for Question 2	**6**	

Question	Scheme	Marks	Notes
3	$\frac{\mathrm{d}r}{\mathrm{d}\theta} = 2\theta$	**B1**	
	$s = \int_0^{\sqrt{5}} \sqrt{\theta^4 + 4\theta^2}\,\mathrm{d}\theta$	**M1**	Uses $\sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}$ with $r = \theta^2$ and their derivative. Condone missing limits.
	$= \int_0^{\sqrt{5}} \theta\sqrt{\theta^2 + 4}\,\mathrm{d}\theta$	**A1**	$\theta\sqrt{\theta^2+4}$ oe, with θ taken out of the root; limits need not be seen.
	$u = \theta^2 + 4$ $s = \int_4^9 \frac{1}{2}u^{\frac{1}{2}}\,\mathrm{d}u$ $= \left[\frac{1}{3}u^{\frac{3}{2}}\right]_4^9 = \ldots$	**dM1**	Integrates to the form $k\left(\theta^2+4\right)^{\frac{3}{2}}$, or $ku^{\frac{3}{2}}$, and applies limits matching their variable. Condone a wrong k.
	$s = \frac{19}{3}$	**A1**	oe exact.
	Alternative: Parametric form		
	$x = \theta^2\cos\theta,\ y = \theta^2\sin\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\theta\cos\theta - \theta^2\sin\theta$ $\frac{\mathrm{d}y}{\mathrm{d}\theta} = 2\theta\sin\theta + \theta^2\cos\theta$	**M1**	Writes $x = r\cos\theta$, $y = r\sin\theta$ with $r = \theta^2$ and differentiates both by the product rule, two terms each. Condone sign and coefficient slips.
	$\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2$ $= \theta^4 + 4\theta^2$ $s = \int_0^{\sqrt{5}} \sqrt{\theta^4 + 4\theta^2}\,\mathrm{d}\theta$	**M1**	Squares and adds their derivatives in the parametric arc length integral, using $\sin^2\theta + \cos^2\theta = 1$ on the squared terms. Condone missing limits, and cross terms left by a slip.
	$= \int_0^{\sqrt{5}} \theta\sqrt{\theta^2 + 4}\,\mathrm{d}\theta$	**A1**	$\theta\sqrt{\theta^2+4}$ oe, with θ taken out of the root; limits need not be seen.
	$u = \theta^2 + 4$ $s = \int_4^9 \frac{1}{2}u^{\frac{1}{2}}\,\mathrm{d}u$ $= \left[\frac{1}{3}u^{\frac{3}{2}}\right]_4^9 = \ldots$	**dM1**	Integrates to the form $k\left(\theta^2+4\right)^{\frac{3}{2}}$, or $ku^{\frac{3}{2}}$, and applies limits matching their variable. Condone a wrong k.
	$s = \frac{19}{3}$	**A1**	oe exact.
	Total for Question 3	**5**	

Question	Scheme	Marks	Notes
4	$\frac{\mathrm{d}x}{\mathrm{d}t} = 4t^2 - \frac{1}{t}$, $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{3k}{2}t^{\frac{1}{2}}$	**M1 A1**	**M1** Differentiates x and y with respect to t, with the power of at least one term decreased by 1. **A1** Both correct.
	$\left(4t^2 - \frac{1}{t}\right)^2 + \frac{9k^2}{4}t = \left(4t^2 + \frac{1}{t}\right)^2$ $\Rightarrow -8t + \frac{9k^2}{4}t = 8t \Rightarrow k^2 = \ldots$	**M1**	Squares and adds their derivatives, sets equal to the given expression and compares coefficients of t or substitutes a value of t. Condone slips.
	$k = \frac{8}{3}$	**A1**	Positive value only.
	$s = \int_1^2 \left(4t^2 + \frac{1}{t}\right) \mathrm{d}t$ $= \left[\frac{4t^3}{3} + \ln t\right]_1^2$	**M1**	Uses the arc length formula, takes the square root of the given expression and integrates, with at least one term correct.
	$= \left(\frac{32}{3} + \ln 2\right) - \left(\frac{4}{3} + \ln 1\right)$	**dM1**	Substitutes $t = 2$ and $t = 1$ into their integral and subtracts.
	$= \frac{28}{3} + \ln 2$	**A1**	cao, in the form $p + \ln q$ with $p = \frac{28}{3}$ oe and $q = 2$.
	Total for Question 4	**7**	

Question	Scheme	Marks	Notes
5(a)	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{1}{2}\operatorname{sech}^2\frac{x}{2}}{\tanh\frac{x}{2}}$	**M1**	Differentiates by the chain rule to the form $\frac{k\operatorname{sech}^2\frac{x}{2}}{\tanh\frac{x}{2}}$, or after splitting the logarithm, or in exponentials. Condone a wrong constant k.
	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2\sinh\frac{x}{2}\cosh\frac{x}{2}}$ $= \frac{1}{\sinh x} = \operatorname{cosech} x$	**A1 AG**	Correct derivative leading without error to $\operatorname{cosech} x$, by the double angle formula or exponentials.
		[2]	
5(b)	$s = \int_a^b \sqrt{1+\operatorname{cosech}^2 x}\,\mathrm{d}x$ $1+\operatorname{cosech}^2 x = \coth^2 x$	**M1**	Substitutes $\operatorname{cosech} x$ into the arc length formula and simplifies $1+\operatorname{cosech}^2 x$ by an identity or exponentials. Condone sign slips in the identity.
	$s = \int_a^b \coth x\,\mathrm{d}x$	**A1**	Integrand $\coth x$ or $\frac{\cosh x}{\sinh x}$ oe.
	$s = [\ln(\sinh x)]_a^b$ $= \ln(\sinh b) - \ln(\sinh a)$	**dM1**	Integrates their $\coth x$ to $k\ln(\sinh x)$, or an equivalent form, and substitutes a and b.
	$s = \ln\left(\frac{\sinh b}{\sinh a}\right)$	**A1 AG**	Correct integral and limits leading without error to the printed result; $\ln\lvert\sinh x\rvert$ accepted.
		[4]	
	Total for Question 5	**6**	

Question	Scheme	Marks	Notes
6	$\frac{\mathrm{d}x}{\mathrm{d}t} = 2\cosh t, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 2\sinh 2t$	**B1**	Both correct, oe such as $4\sinh t\cosh t$.
	$s = \int \sqrt{\text{“}4\cosh^2 t\text{”} + \text{“}4\sinh^2 2t\text{”}}\,\mathrm{d}t$	**M1**	Uses the parametric formula with their derivatives. Condone coefficient slips.
	$s = \int_0^{\ln 2} 2\cosh t\sqrt{1+4\sinh^2 t}\,\mathrm{d}t$	**A1**	Correct integral with $\cosh t$ taken out of the root, oe, with limits 0 and $\ln 2$.
	$u = 2\sinh t \Rightarrow \mathrm{d}u = 2\cosh t\,\mathrm{d}t$ $t = \ln 2 \Rightarrow u = 2\sinh(\ln 2) = \ldots$	**M1**	Replaces $\sinh t$ and $\mathrm{d}t$ in their integral using $u = 2\sinh t$, and changes both limits. Condone slips evaluating $\sinh(\ln 2)$.
	$\sinh(\ln 2) = \frac{2-\frac{1}{2}}{2} = \frac{3}{4} \Rightarrow u = \frac{3}{2}$ $s = \int_0^{\frac{3}{2}} \sqrt{1+u^2}\,\mathrm{d}u$	**A1 AG**	Correct substituted integral reached without error, with $\sinh(\ln 2)$ evaluated from exponentials, not stated, to give the upper limit $\frac{3}{2}$.
	Total for Question 6	**5**	

Question	Scheme	Marks	Notes
7	$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\frac{5}{3}e^{-\frac{5}{12}\theta}$	**B1**	
	$s = \int_0^\beta \sqrt{16e^{-\frac{5}{6}\theta} + \frac{25}{9}e^{-\frac{5}{6}\theta}}\,\mathrm{d}\theta$	**M1**	Uses $\sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}$ with r and their $\frac{\mathrm{d}r}{\mathrm{d}\theta}$, squaring both and combining into one exponential term. Condone coefficient slips.
	$= \int_0^\beta \frac{13}{3}e^{-\frac{5}{12}\theta}\,\mathrm{d}\theta$	**A1**	Correct integrand $\frac{13}{3}e^{-\frac{5}{12}\theta}$ oe; the root must be taken.
	$s = \left[-\frac{52}{5}e^{-\frac{5}{12}\theta}\right]_0^\beta$ $= \frac{52}{5}\left(1 - e^{-\frac{5}{12}\beta}\right)$ $\Rightarrow -\frac{5}{12}\beta = \ln\left(1 - \frac{5s}{52}\right)$ $\Rightarrow \beta = \ldots$	**dM1**	Integrates to the form $ke^{-\frac{5}{12}\theta}$, applies limits 0 and β, and takes logarithms to make β the subject. Condone sign slips.
	$\beta = \frac{12}{5}\ln\left(\frac{52}{52-5s}\right)$	**A1**	oe, such as $-\frac{12}{5}\ln\left(1 - \frac{5s}{52}\right)$.
	$0 < s < \frac{52}{5}$	**B1**	Both ends with strict inequalities, oe such as $0 < s < 10.4$.
	Alternative: Parametric form		
	$x = 4e^{-\frac{5}{12}\theta}\cos\theta,\ y = 4e^{-\frac{5}{12}\theta}\sin\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\frac{1}{3}e^{-\frac{5}{12}\theta}(5\cos\theta + 12\sin\theta)$ $\frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{1}{3}e^{-\frac{5}{12}\theta}(12\cos\theta - 5\sin\theta)$	**M1**	Writes $x = r\cos\theta$, $y = r\sin\theta$ and differentiates both by the product rule, two terms each. Condone sign and coefficient slips.
	$s = \int_0^\beta \sqrt{\frac{169}{9}e^{-\frac{5}{6}\theta}\left(\cos^2\theta + \sin^2\theta\right)}\,\mathrm{d}\theta$	**M1**	Uses $\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}$ with their derivatives, using $\cos^2\theta + \sin^2\theta = 1$ on the squared terms. Condone coefficient slips, and cross terms left by a slip.
	$= \int_0^\beta \frac{13}{3}e^{-\frac{5}{12}\theta}\,\mathrm{d}\theta$	**A1**	Correct integrand $\frac{13}{3}e^{-\frac{5}{12}\theta}$ oe; the root must be taken.
	$s = \left[-\frac{52}{5}e^{-\frac{5}{12}\theta}\right]_0^\beta$ $= \frac{52}{5}\left(1 - e^{-\frac{5}{12}\beta}\right)$ $\Rightarrow -\frac{5}{12}\beta = \ln\left(1 - \frac{5s}{52}\right)$ $\Rightarrow \beta = \ldots$	**dM1**	Integrates to the form $ke^{-\frac{5}{12}\theta}$, applies limits 0 and β, and takes logarithms to make β the subject. Condone sign slips.
	$\beta = \frac{12}{5}\ln\left(\frac{52}{52-5s}\right)$	**A1**	oe, such as $-\frac{12}{5}\ln\left(1 - \frac{5s}{52}\right)$.
	$0 < s < \frac{52}{5}$	**B1**	Both ends with strict inequalities, oe such as $0 < s < 10.4$.
	Total for Question 7	**6**	

Question	Scheme	Marks	Notes
8(a)	$y = \frac{1}{\cos 2x} = \sec 2x$	**B1**	Uses $\cos^2 x - \sin^2 x = \cos 2x$. May be implied by later working in $2x$.
	$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sec 2x \tan 2x$	**M1 A1**	**M1** Differentiates to the form $k\sec 2x \tan 2x$, or an equivalent such as $\frac{k\sin 2x}{\cos^2 2x}$ by the chain or quotient rule. Condone a wrong k. **A1** Correct derivative, oe.
	$1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1 + 4\sec^2 2x \tan^2 2x$ $= 1 + 4\sec^2 2x \left(\sec^2 2x - 1\right)$	**M1**	Substitutes their $\frac{\mathrm{d}y}{\mathrm{d}x}$ into $1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$ and uses a Pythagorean identity to write it in one trigonometric function. Condone slips.
	$= \left(2\sec^2 2x - 1\right)^2$	**A1**	Correct perfect square, oe such as $\frac{\left(2-\cos^2 2x\right)^2}{\cos^4 2x}$.
	$s = \int_0^{\frac{\pi}{6}} \sqrt{\left(2\sec^2 2x - 1\right)^2}\,\mathrm{d}x$ $= \int_0^{\frac{\pi}{6}} \left(2\sec^2 2x - 1\right)\mathrm{d}x$	**A1 AG**	Arc length integral with limits 0 and $\frac{\pi}{6}$ and the correct perfect square, leading without error to the printed result. Positivity of $2\sec^2 2x - 1$ need not be stated.
		[6]	
8(b)	$s = [\tan 2x - x]_0^{\frac{\pi}{6}}$	**M1 A1**	**M1** Integrates to the form $p\tan 2x + qx$, $p \neq 0$. **A1** Correct integral, oe.
	$\left(\tan\frac{\pi}{3} - \frac{\pi}{6}\right) - (\tan 0 - 0)$	**dM1**	Substitutes both limits into their $p\tan 2x + qx$ and subtracts. The zero from $x = 0$ may be implied.
	$s = \sqrt{3} - \frac{\pi}{6}$	**A1**	Exact, oe such as $\frac{6\sqrt{3}-\pi}{6}$.
		[4]	
	Total for Question 8	**10**	

Question	Scheme	Marks	Notes
9(a)	$\frac{\mathrm{d}y}{\mathrm{d}x} = 3 \times \frac{2x}{x^2-9}$	**M1**	Differentiates to the form $\frac{kx}{x^2-9}$.
	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6x}{x^2-9}$	**A1**	
	$s = \int_4^5 \sqrt{1 + \left(\text{“}\frac{6x}{x^2-9}\text{”}\right)^2}\,\mathrm{d}x$ $= \int_4^5 \sqrt{\frac{(x^2-9)^2+36x^2}{(x^2-9)^2}}\,\mathrm{d}x$	**M1**	Substitutes their $\frac{\mathrm{d}y}{\mathrm{d}x}$ into the arc length formula and combines $1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$ over a common denominator. Condone expansion slips.
	$= \int_4^5 \sqrt{\frac{(x^2+9)^2}{(x^2-9)^2}}\,\mathrm{d}x$ $= \int_4^5 \frac{x^2+9}{x^2-9}\,\mathrm{d}x$	**A1 AG**	Correct substituted formula leading without error, via the perfect square $\left(x^2+9\right)^2$, to the printed integral with its limits. Combined steps accepted.
		[4]	
9(b)	$\frac{x^2+9}{x^2-9} = 1 + \frac{18}{x^2-9}$	**B1**	The constant 1, by division or as the constant term of a partial fractions form for the whole expression.
	$\frac{18}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$	**M1**	Sets up fractions over $x-3$ and $x+3$ for their remaining fraction, or with a constant term for the whole expression.
	$18 = A(x+3) + B(x-3) \Rightarrow A = \ldots,\ B = \ldots$	**dM1**	Finds both constants by substitution or equating coefficients. Condone arithmetic slips.
	$1 + \frac{3}{x-3} - \frac{3}{x+3}$	**A1**	Full expression, including the 1, oe.
		[4]	
9(c)	$\int_4^5 \left(1 + \frac{3}{x-3} - \frac{3}{x+3}\right)\mathrm{d}x$ $= [x + 3\ln(x-3) - 3\ln(x+3)]_4^5$	**M1 A1**	**M1** Integrates their partial fractions, with at least one term of the form $k\ln(x \pm 3)$. Condone coefficient slips. **A1** Correct integral; modulus signs accepted.
	$(5 + 3\ln 2 - 3\ln 8) - (4 + 3\ln 1 - 3\ln 7)$ $= 1 + 3\ln\left(\frac{2\times 7}{8}\right)$	**dM1**	Substitutes 5 and 4 into their integral, subtracts, and combines their logarithms using the laws of logarithms.

Question 9 continued

Question	Scheme	Marks	Notes
	$1 + 3\ln\frac{7}{4}$	**A1**	Any form $p + q\ln r$ with integers p, q and rational r, such as $1 + \ln\frac{343}{64}$.
		[4]	
	Total for Question 9	**12**	

Question	Scheme	Marks	Notes
10	$\frac{\mathrm{d}x}{\mathrm{d}t} = t^2 - (\ln t)^2 - 2\ln t + 2\ln t + 2 - 2$ $\frac{\mathrm{d}y}{\mathrm{d}t} = 2t\ln t + t - t$	**M1**	Differentiates both x and y, using the product rule on at least one product of a power of t and $\ln t$. Condone sign and coefficient slips.
	$\frac{\mathrm{d}x}{\mathrm{d}t} = t^2 - (\ln t)^2$ $\frac{\mathrm{d}y}{\mathrm{d}t} = 2t\ln t$	**A1**	Both correct and simplified.
	$s = \int_1^e \sqrt{(\text{“}t^2 - (\ln t)^2\text{”})^2 + (\text{“}2t\ln t\text{”})^2}\,\mathrm{d}t$ $t^4 + 2t^2(\ln t)^2 + (\ln t)^4 = \left(t^2 + (\ln t)^2\right)^2$	**M1**	Squares and adds their derivatives under a root in the arc length integral, then expands and attempts a perfect square. Limits need not be seen.
	$s = \int_1^e \left(t^2 + (\ln t)^2\right)\mathrm{d}t$	**A1**	Correct integrand from correct derivatives. Limits need not be seen.
	$\int (\ln t)^2\,\mathrm{d}t = t(\ln t)^2 - \int t \times \frac{2\ln t}{t}\,\mathrm{d}t$	**M1**	Applies integration by parts to $(\ln t)^2$, reaching the form $pt(\ln t)^2 - q\int \ln t\,\mathrm{d}t$, oe after substituting $t = e^w$. Condone coefficient slips.
	$\int (\ln t)^2\,\mathrm{d}t = t(\ln t)^2 - 2t\ln t + 2t$	**A1**	Oe. The constant need not be seen.
	$s = \left[\frac{1}{3}t^3 + t(\ln t)^2 - 2t\ln t + 2t\right]_1^e$ $= \left(\frac{1}{3}e^3 + e - 2e + 2e\right) - \left(\frac{1}{3} + 2\right)$	**dM1**	Integrates t^2 and substitutes $t = e$ and $t = 1$ into their whole antiderivative, subtracting.
	$s = \frac{1}{3}e^3 + e - \frac{7}{3}$	**A1**	Any exact equivalent, such as $\frac{1}{3}\left(e^3 + 3e - 7\right)$.
	Total for Question 10	**8**	

Question	Scheme	Marks	Notes
11(a)	$y^2 = 2(y+4) \Rightarrow y^2 - 2y - 8 = 0$ $\Rightarrow y = \ldots$	**M1**	Eliminates x or y between ℓ and C and solves the resulting quadratic by any means.
	$P(1,-2)$, $Q(4,4)$	**A1**	Both points; the y-values imply $\gamma = -2$, $\beta = 4$.
	$PQ = \sqrt{(4-1)^2 + (4+2)^2}$ $x = \frac{y^2}{4} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{y}{2}$ $PQ + \int_{-2}^{4} \sqrt{1 + \left(\frac{y}{2}\right)^2}\,\mathrm{d}y$	**M1**	Adds a distance-formula chord between their P and Q to an arc length integral in y using their $\frac{\mathrm{d}x}{\mathrm{d}y}$. Condone slips in the chord.
	$3\sqrt{5} + \int_{-2}^{4} \sqrt{1 + \frac{y^2}{4}}\,\mathrm{d}y$ $\alpha = 3\sqrt{5},\ \beta = 4,\ \gamma = -2$	**A1**	cso. Complete expression or all three constants, $\alpha = 3\sqrt{5}$ oe such as $\sqrt{45}$, with $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{y}{2}$ oe seen.
		[4]	
11(b)	$\mathrm{d}y = 2\cosh u\,\mathrm{d}u$ $\int \sqrt{1 + \sinh^2 u} \times 2\cosh u\,\mathrm{d}u$	**M1**	Substitutes for y and replaces $\mathrm{d}y$ by $k\cosh u\,\mathrm{d}u$.
	$= \int 2\cosh^2 u\,\mathrm{d}u$	**A1**	
	$y = 4 \Rightarrow u = \ln\left(2+\sqrt{5}\right)$ $y = -2 \Rightarrow u = \ln\left(\sqrt{2}-1\right)$	**B1**	Both limits, as logarithms or arsinh 2, arsinh(-1); or a correct substitution back to y with limits -2 and 4.
	$\int (1 + \cosh 2u)\,\mathrm{d}u$	**M1**	Uses a double-angle identity or exponentials for $\cosh^2 u$ and integrates to the form $pu + q\sinh 2u$ oe. Condone sign and coefficient slips.
	$= \left[u + \frac{1}{2}\sinh 2u\right]$	**A1**	Correct, oe such as $u + \sinh u \cosh u$.
	$\left(\ln\left(2+\sqrt{5}\right) + 2\sqrt{5}\right)$ $-\left(\ln\left(\sqrt{2}-1\right) - \sqrt{2}\right) +$ "$3\sqrt{5}$"	**dM1**	Substitutes both their u limits, evaluating $\sinh 2u$ or $\sinh u \cosh u$ at each, subtracts and adds their α.
	$5\sqrt{5} + \sqrt{2} + \ln\left(\left(2+\sqrt{5}\right)\left(1+\sqrt{2}\right)\right)$	**A1**	cso. Exact, surds collected and one logarithm, oe such as $\ln\left(2 + 2\sqrt{2} + \sqrt{5} + \sqrt{10}\right)$.
		[7]	
	Total for Question 11	**11**	

Question	Scheme	Marks	Notes
12(a)	$\frac{\mathrm{d}y}{\mathrm{d}x} = 8x^{-\frac{1}{2}}$	**B1**	Oe, such as $\frac{8}{\sqrt{x}}$.
	$s = \int_{36}^{225} \sqrt{1 + \left(\text{“}\frac{8}{\sqrt{x}}\text{”}\right)^2}\,\mathrm{d}x$ $1 + \frac{64}{x} = \frac{x+64}{x}$	**M1**	Uses the arc length formula with their squared derivative and writes 1+ their $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$ as a single fraction.
	$s = \int_{36}^{225} \sqrt{\frac{x+64}{x}}\,\mathrm{d}x$	**A1 AG**	Correct formula with limits 36 and 225 leading without error to the printed integral.
		[3]	
12(b)(i)	$\operatorname{arsinh}\frac{15}{8} = \ln\left(\frac{15}{8} + \sqrt{\frac{225}{64}+1}\right)$	**M1**	Uses $\operatorname{arsinh} t = \ln\left(t + \sqrt{t^2+1}\right)$ on at least one value. Condone slips in the square root.
	$\operatorname{arsinh}\frac{15}{8} = \ln 4$, $\operatorname{arsinh}\frac{3}{4} = \ln 2$	**A1**	Both.
	$\ln 4 = 2\ln 2 \Rightarrow \operatorname{arsinh}\frac{15}{8} = 2\operatorname{arsinh}\frac{3}{4}$	**A1 AG**	Both values correct, leading without error via $\ln 4 = 2\ln 2$ oe to a statement of the printed result.
		[3]	
12(b)(ii)	$\mathrm{d}x = 128\sinh\theta\cosh\theta\,\mathrm{d}\theta$ $s = \int \sqrt{\frac{64\sinh^2\theta+64}{64\sinh^2\theta}} \times 128\sinh\theta\cosh\theta\,\mathrm{d}\theta$	**M1**	Substitutes for x and replaces $\mathrm{d}x$ by $k\sinh\theta\cosh\theta\,\mathrm{d}\theta$. Condone coefficient slips.
	$s = \int_{\ln 2}^{\ln 4} 128\cosh^2\theta\,\mathrm{d}\theta$	**A1**	Correct integrand; limits need not be shown.
	$\int 64(1+\cosh 2\theta)\,\mathrm{d}\theta = \ldots$	**M1**	Uses a double angle identity, or exponentials, and integrates to the form $a\theta + b\sinh 2\theta$ oe. Condone sign and coefficient slips.
	$[64\theta + 32\sinh 2\theta]_{\ln 2}^{\ln 4}$	**A1**	Oe, such as $64\theta + 64\sinh\theta\cosh\theta$.
	$64(\ln 4 - \ln 2)$ $+ 64\left(\frac{15}{8}\times\frac{17}{8} - \frac{3}{4}\times\frac{5}{4}\right)$	**dM1**	Substitutes their θ limits, $\ln 2$ and $\ln 4$ oe, or $x = 36$ and $x = 225$ after substituting back, into their integral, subtracts, and evaluates the hyperbolic terms, by $\sinh\theta$ and $\cosh\theta$ values or exponentials.
	$s = 195 + 64\ln 2$	**A1**	Or $p = 195$, $q = 64$.
		[6]	
	Total for Question 12	**12**	