Questions

1. The infinite series C and S are defined by

$$C = \cos 2\theta - \frac{1}{3}\cos 5\theta + \frac{1}{9}\cos 8\theta - \frac{1}{27}\cos 11\theta + \cdots$$
$$S = \sin 2\theta - \frac{1}{3}\sin 5\theta + \frac{1}{9}\sin 8\theta - \frac{1}{27}\sin 11\theta + \cdots$$

Given that the series C and S are both convergent.

(a) Show that

$$C + \mathrm{i}S = \frac{3e^{2\mathrm{i}\theta}}{3 + e^{3\mathrm{i}\theta}}$$ **[4]**

(b) Hence show that

$$S = \frac{9\sin 2\theta - 3\sin\theta}{10 + 6\cos 3\theta}$$ **[4]**

Solution

(a) Using Euler's formula,

$$\cos x + \mathrm{i}\sin x = e^{\mathrm{i}x}$$

So

$$\begin{aligned} C + \mathrm{i}S &= (\cos 2\theta + \mathrm{i}\sin 2\theta) - \frac{1}{3}(\cos 5\theta + \mathrm{i}\sin 5\theta) + \frac{1}{9}(\cos 8\theta + \mathrm{i}\sin 8\theta) - \cdots \\ &= e^{2\mathrm{i}\theta} - \frac{1}{3}e^{5\mathrm{i}\theta} + \frac{1}{9}e^{8\mathrm{i}\theta} - \frac{1}{27}e^{11\mathrm{i}\theta} + \cdots \end{aligned}$$

Factor out $e^{2\mathrm{i}\theta}$:

$$C + \mathrm{i}S = e^{2\mathrm{i}\theta}\left(1 - \frac{1}{3}e^{3\mathrm{i}\theta} + \frac{1}{9}e^{6\mathrm{i}\theta} - \frac{1}{27}e^{9\mathrm{i}\theta} + \cdots\right)$$

This is a geometric series with first term 1 and common ratio

$$r = -\frac{e^{3\mathrm{i}\theta}}{3}$$

Since

$$|r| = \frac{|e^{3\mathrm{i}\theta}|}{3} = \frac{1}{3} < 1$$

the series converges, so

$$\begin{aligned} C + \mathrm{i}S &= e^{2\mathrm{i}\theta} \cdot \frac{1}{1 - (-e^{3\mathrm{i}\theta}/3)} \\ &= e^{2\mathrm{i}\theta} \cdot \frac{1}{1 + e^{3\mathrm{i}\theta}/3} \\ &= \frac{3e^{2\mathrm{i}\theta}}{3 + e^{3\mathrm{i}\theta}} \end{aligned}$$

Hence

$$C + \mathrm{i}S = \frac{3e^{2\mathrm{i}\theta}}{3 + e^{3\mathrm{i}\theta}}$$

(b) From part (a),

$$C + \mathrm{i}S = \frac{3e^{2\mathrm{i}\theta}}{3 + e^{3\mathrm{i}\theta}}$$

Multiply numerator and denominator by $3 + e^{-3i\theta}$:

$$C + iS = \frac{3e^{2i\theta}(3 + e^{-3i\theta})}{(3 + e^{3i\theta})(3 + e^{-3i\theta})}$$

$$= \frac{9e^{2i\theta} + 3e^{-i\theta}}{9 + 3e^{3i\theta} + 3e^{-3i\theta} + e^{3i\theta}e^{-3i\theta}}$$

$$= \frac{9e^{2i\theta} + 3e^{-i\theta}}{10 + 3\left(e^{3i\theta} + e^{-3i\theta}\right)}$$

Now use

$$e^{3i\theta} + e^{-3i\theta} = 2\cos 3\theta$$

so the denominator becomes

$$10 + 6\cos 3\theta$$

Therefore

$$C + iS = \frac{9e^{2i\theta} + 3e^{-i\theta}}{10 + 6\cos 3\theta}$$

The denominator is real, so taking imaginary parts gives

$$S = \operatorname{Im}\left[\frac{9e^{2i\theta} + 3e^{-i\theta}}{10 + 6\cos 3\theta}\right]$$

$$= \frac{9\sin 2\theta + 3(-\sin\theta)}{10 + 6\cos 3\theta}$$

Hence

$$S = \frac{9\sin 2\theta - 3\sin\theta}{10 + 6\cos 3\theta}$$

2. (i) Show that

$$(1-2e^{2i\theta})(1-2e^{-2i\theta}) = 5 - 4\cos 2\theta \qquad \textbf{[3]}$$

For a positive integer n, series C and S are defined by

$$C = \cos\theta + 2\cos 3\theta + 4\cos 5\theta + \cdots + 2^{n-1}\cos((2n-1)\theta)$$
$$S = \sin\theta + 2\sin 3\theta + 4\sin 5\theta + \cdots + 2^{n-1}\sin((2n-1)\theta)$$

(ii) Show that

$$C = \frac{2^{n+1}\cos((2n-1)\theta) - 2^n\cos((2n+1)\theta) - \cos\theta}{5-4\cos 2\theta}$$

and find a similar expression for S. **[9]**

Solution

(i) Expanding the product,

$$\begin{aligned}(1-2e^{2i\theta})(1-2e^{-2i\theta}) &= 1 - 2e^{2i\theta} - 2e^{-2i\theta} + 4 \\ &= 5 - 2\left(e^{2i\theta} + e^{-2i\theta}\right)\end{aligned}$$

Using

$$e^{2i\theta} + e^{-2i\theta} = 2\cos 2\theta$$

gives

$$\begin{aligned}(1-2e^{2i\theta})(1-2e^{-2i\theta}) &= 5 - 2(2\cos 2\theta) \\ &= 5 - 4\cos 2\theta\end{aligned}$$

as required.

(ii) Let

$$Z = C + iS$$

Then, using $\cos x + i\sin x = e^{ix}$,

$$\begin{aligned}Z &= (\cos\theta + i\sin\theta) + 2(\cos 3\theta + i\sin 3\theta) + \cdots + 2^{n-1}(\cos((2n-1)\theta) + i\sin((2n-1)\theta)) \\ &= e^{i\theta} + 2e^{3i\theta} + 4e^{5i\theta} + \cdots + 2^{n-1}e^{(2n-1)i\theta} \\ &= e^{i\theta}\left(1 + 2e^{2i\theta} + (2e^{2i\theta})^2 + \cdots + (2e^{2i\theta})^{n-1}\right)\end{aligned}$$

This is a geometric series with first term 1 and common ratio $2e^{2i\theta}$, so

$$Z = e^{i\theta}\frac{1-(2e^{2i\theta})^n}{1-2e^{2i\theta}} = e^{i\theta}\frac{1-2^n e^{2ni\theta}}{1-2e^{2i\theta}}$$

Multiply numerator and denominator by $1-2e^{-2i\theta}$:

$$\begin{aligned}Z &= e^{i\theta}\frac{1-2^n e^{2ni\theta}}{1-2e^{2i\theta}} \cdot \frac{1-2e^{-2i\theta}}{1-2e^{-2i\theta}} \\ &= \frac{e^{i\theta}(1-2^n e^{2ni\theta})(1-2e^{-2i\theta})}{(1-2e^{2i\theta})(1-2e^{-2i\theta})}\end{aligned}$$

Using part (i), the denominator is $5-4\cos 2\theta$, so

$$Z = \frac{e^{i\theta}(1-2^n e^{2ni\theta})(1-2e^{-2i\theta})}{5-4\cos 2\theta}$$

$$= \frac{e^{i\theta} - 2e^{-i\theta} - 2^n e^{i(2n+1)\theta} + 2^{n+1}e^{i(2n-1)\theta}}{5-4\cos 2\theta}$$

Now simplify the first two terms:

$$\begin{aligned} e^{\mathrm{i}\theta} - 2e^{-\mathrm{i}\theta} &= (\cos\theta + \mathrm{i}\sin\theta) - 2(\cos\theta - \mathrm{i}\sin\theta) \\ &= -\cos\theta + 3\mathrm{i}\sin\theta \end{aligned}$$

Hence

$$Z = \frac{-\cos\theta + 3\mathrm{i}\sin\theta - 2^n e^{\mathrm{i}(2n+1)\theta} + 2^{n+1} e^{\mathrm{i}(2n-1)\theta}}{5 - 4\cos 2\theta}$$

Writing the exponential terms in sine and cosine form,

$$\begin{aligned} Z = &\frac{-\cos\theta - 2^n\cos((2n+1)\theta) + 2^{n+1}\cos((2n-1)\theta)}{5 - 4\cos 2\theta} \\ &+ \mathrm{i}\,\frac{3\sin\theta - 2^n\sin((2n+1)\theta) + 2^{n+1}\sin((2n-1)\theta)}{5 - 4\cos 2\theta} \end{aligned}$$

Since $Z = C + \mathrm{i}S$, equating real and imaginary parts gives

$$C = \frac{2^{n+1}\cos((2n-1)\theta) - 2^n\cos((2n+1)\theta) - \cos\theta}{5 - 4\cos 2\theta}$$

and

$$S = \frac{2^{n+1}\sin((2n-1)\theta) - 2^n\sin((2n+1)\theta) + 3\sin\theta}{5 - 4\cos 2\theta}$$

So the required similar expression for S is

$$S = \frac{2^{n+1}\sin((2n-1)\theta) - 2^n\sin((2n+1)\theta) + 3\sin\theta}{5 - 4\cos 2\theta}$$

3. **(i)** Show that

$$1 - e^{i2\theta} = 2\sin\theta(\sin\theta - i\cos\theta)$$ **[3]**

(ii) For a positive integer n, the series C and S are defined as follows.

$$C = 1 - \binom{n}{1}\cos 2\theta + \binom{n}{2}\cos 4\theta - \ldots + (-1)^n \cos 2n\theta$$

$$S = \quad -\binom{n}{1}\sin 2\theta + \binom{n}{2}\sin 4\theta - \ldots + (-1)^n \sin 2n\theta$$

By considering $C + iS$, show that

$$C = 2^n \sin^n\theta \cos\left(n\theta - \frac{n\pi}{2}\right)$$

and find a corresponding expression for S. **[9]**

Solution

(i) Using Euler's formula $e^{i2\theta} = \cos 2\theta + i\sin 2\theta$,

$$\begin{aligned} 1 - e^{i2\theta} &= 1 - \cos 2\theta - i\sin 2\theta \\ &= (1 - \cos 2\theta) - i\sin 2\theta \\ &= 2\sin^2\theta - i(2\sin\theta\cos\theta) \\ &= 2\sin\theta(\sin\theta - i\cos\theta) \end{aligned}$$

Hence

$$1 - e^{i2\theta} = 2\sin\theta(\sin\theta - i\cos\theta)$$

(ii) Combine the two series into one complex expression:

$$\begin{aligned} C + iS &= 1 - \binom{n}{1}(\cos 2\theta + i\sin 2\theta) + \binom{n}{2}(\cos 4\theta + i\sin 4\theta) - \cdots + (-1)^n(\cos 2n\theta + i\sin 2n\theta) \\ &= 1 - \binom{n}{1}e^{i2\theta} + \binom{n}{2}e^{i4\theta} - \cdots + (-1)^n e^{i2n\theta} \\ &= \sum_{r=0}^{n}\binom{n}{r}(-1)^r e^{i2r\theta} \\ &= \sum_{r=0}^{n}\binom{n}{r}\left(-e^{i2\theta}\right)^r \\ &= (1 - e^{i2\theta})^n \end{aligned}$$

by the binomial theorem.

From part (i),

$$1 - e^{i2\theta} = 2\sin\theta(\sin\theta - i\cos\theta)$$

so

$$\begin{aligned} C + iS &= [2\sin\theta(\sin\theta - i\cos\theta)]^n \\ &= 2^n\sin^n\theta\,(\sin\theta - i\cos\theta)^n \end{aligned}$$

Now factor out $-i$:

$$\sin\theta - i\cos\theta = -i(\cos\theta + i\sin\theta)$$

Since

$$-i = e^{-i\pi/2}$$

this becomes

$$\sin\theta - i\cos\theta = e^{-i\pi/2}(\cos\theta + i\sin\theta)$$

Therefore, by De Moivre's theorem,

$$\begin{aligned}(\sin\theta - i\cos\theta)^n &= \left(e^{-i\pi/2}(\cos\theta + i\sin\theta)\right)^n \\ &= e^{-in\pi/2}\,e^{in\theta} \\ &= \cos\left(n\theta - \frac{n\pi}{2}\right) + i\sin\left(n\theta - \frac{n\pi}{2}\right)\end{aligned}$$

Substituting this into $C + iS$,

$$C + iS = 2^n \sin^n\theta\left(\cos\left(n\theta - \frac{n\pi}{2}\right) + i\sin\left(n\theta - \frac{n\pi}{2}\right)\right)$$

Equating real and imaginary parts gives

$$C = 2^n \sin^n\theta\cos\left(n\theta - \frac{n\pi}{2}\right)$$

and

$$S = 2^n \sin^n\theta\sin\left(n\theta - \frac{n\pi}{2}\right)$$

4. The infinite series C and S are defined as follows:

$$C = \cos\alpha + r\cos(\alpha+\beta) + r^2\cos(\alpha+2\beta) + \dots$$

$$S = \sin\alpha + r\sin(\alpha+\beta) + r^2\sin(\alpha+2\beta) + \dots$$

where r is a real number and $|r| < 1$.

By considering $C + \mathrm{i}S$, show that

$$C = \frac{\cos\alpha - r\cos(\alpha-\beta)}{1 + r^2 - 2r\cos\beta}$$

and find a corresponding expression for S. **[8]**

Solution

Let

$$Z = C + \mathrm{i}S$$

Then

$$\begin{aligned} Z &= \left(\cos\alpha + r\cos(\alpha+\beta) + r^2\cos(\alpha+2\beta) + \cdots\right) \\ &\qquad + \mathrm{i}\left(\sin\alpha + r\sin(\alpha+\beta) + r^2\sin(\alpha+2\beta) + \cdots\right) \\ &= \sum_{n=0}^{\infty} r^n\left(\cos(\alpha+n\beta) + \mathrm{i}\sin(\alpha+n\beta)\right) \end{aligned}$$

Using Euler's formula,

$$\cos\theta + \mathrm{i}\sin\theta = e^{\mathrm{i}\theta}$$

so

$$\begin{aligned} Z &= \sum_{n=0}^{\infty} r^n e^{\mathrm{i}(\alpha+n\beta)} \\ &= e^{\mathrm{i}\alpha}\sum_{n=0}^{\infty}\left(re^{\mathrm{i}\beta}\right)^n \end{aligned}$$

Since

$$\left|re^{\mathrm{i}\beta}\right| = |r| < 1$$

this is a convergent geometric series, so

$$\begin{aligned} Z &= e^{\mathrm{i}\alpha}\cdot\frac{1}{1 - re^{\mathrm{i}\beta}} \\ &= \frac{e^{\mathrm{i}\alpha}}{1 - re^{\mathrm{i}\beta}} \end{aligned}$$

Now multiply top and bottom by the conjugate-like factor $1 - re^{-\mathrm{i}\beta}$:

$$Z = \frac{e^{\mathrm{i}\alpha}(1 - re^{-\mathrm{i}\beta})}{(1 - re^{\mathrm{i}\beta})(1 - re^{-\mathrm{i}\beta})}$$

First simplify the denominator:

$$\begin{aligned} (1 - re^{\mathrm{i}\beta})(1 - re^{-\mathrm{i}\beta}) &= 1 - r(e^{\mathrm{i}\beta} + e^{-\mathrm{i}\beta}) + r^2 \\ &= 1 - r(2\cos\beta) + r^2 \\ &= 1 + r^2 - 2r\cos\beta \end{aligned}$$

Now simplify the numerator:

$$\begin{aligned} e^{\mathrm{i}\alpha}(1 - re^{-\mathrm{i}\beta}) &= e^{\mathrm{i}\alpha} - re^{\mathrm{i}\alpha}e^{-\mathrm{i}\beta} \\ &= e^{\mathrm{i}\alpha} - re^{\mathrm{i}(\alpha-\beta)} \end{aligned}$$

Hence

$$Z = \frac{e^{\mathrm{i}\alpha} - re^{\mathrm{i}(\alpha-\beta)}}{1 + r^2 - 2r\cos\beta}$$

Write the exponentials in terms of sine and cosine:

$$\begin{aligned} Z &= \frac{\cos\alpha + \mathrm{i}\sin\alpha - r\left(\cos(\alpha-\beta) + \mathrm{i}\sin(\alpha-\beta)\right)}{1 + r^2 - 2r\cos\beta} \\ &= \frac{\left(\cos\alpha - r\cos(\alpha-\beta)\right) + \mathrm{i}\left(\sin\alpha - r\sin(\alpha-\beta)\right)}{1 + r^2 - 2r\cos\beta} \end{aligned}$$

But $Z = C + \mathrm{i}S$, so equating real and imaginary parts gives

$$C = \frac{\cos\alpha - r\cos(\alpha-\beta)}{1 + r^2 - 2r\cos\beta}$$

and

$$S = \frac{\sin\alpha - r\sin(\alpha-\beta)}{1 + r^2 - 2r\cos\beta}$$

Therefore,

$$C = \frac{\cos\alpha - r\cos(\alpha-\beta)}{1 + r^2 - 2r\cos\beta}, \qquad S = \frac{\sin\alpha - r\sin(\alpha-\beta)}{1 + r^2 - 2r\cos\beta}$$

5. (i) For values of θ for which the expressions are defined, show that

$$1 + \mathrm{i}\tan\theta = \sec\theta(\cos\theta + \mathrm{i}\sin\theta)$$ **[2]**

(ii) For a positive integer n, the series C and S are defined by

$$C = 1 - \binom{n}{2}\tan^2\theta + \binom{n}{4}\tan^4\theta - \cdots$$

$$S = \binom{n}{1}\tan\theta - \binom{n}{3}\tan^3\theta + \binom{n}{5}\tan^5\theta - \cdots$$

where in each case the pattern continues until the power of $\tan\theta$ is at most n.

By considering $C + \mathrm{i}S$, show that

$$C = \sec^n\theta\cos n\theta$$

and find a corresponding expression for S. **[7]**

Solution

(i) For the expressions to be defined, we need

$$\cos\theta \neq 0$$

Now

$$\begin{aligned}\sec\theta(\cos\theta + \mathrm{i}\sin\theta) &= \sec\theta\cos\theta + \mathrm{i}\sec\theta\sin\theta \\ &= 1 + \mathrm{i}\tan\theta\end{aligned}$$

So

$$1 + \mathrm{i}\tan\theta = \sec\theta(\cos\theta + \mathrm{i}\sin\theta)$$

as required.

(ii) Consider $C + \mathrm{i}S$.
Using the given definitions,

$$C + \mathrm{i}S = \left(1 - \binom{n}{2}\tan^2\theta + \binom{n}{4}\tan^4\theta - \cdots\right) + \mathrm{i}\left(\binom{n}{1}\tan\theta - \binom{n}{3}\tan^3\theta + \binom{n}{5}\tan^5\theta - \cdots\right)$$

Now note that

$$(\mathrm{i}\tan\theta)^0 = 1, \qquad (\mathrm{i}\tan\theta)^1 = \mathrm{i}\tan\theta, \qquad (\mathrm{i}\tan\theta)^2 = -\tan^2\theta, \qquad (\mathrm{i}\tan\theta)^3 = -\mathrm{i}\tan^3\theta$$

so the terms in $C + \mathrm{i}S$ match the binomial expansion of $(1 + \mathrm{i}\tan\theta)^n$. Hence

$$C + \mathrm{i}S = \sum_{k=0}^{n}\binom{n}{k}(\mathrm{i}\tan\theta)^k = (1 + \mathrm{i}\tan\theta)^n$$

Using part (i),

$$1 + \mathrm{i}\tan\theta = \sec\theta(\cos\theta + \mathrm{i}\sin\theta)$$

Therefore

$$\begin{aligned}C + \mathrm{i}S &= (1 + \mathrm{i}\tan\theta)^n \\ &= [\sec\theta(\cos\theta + \mathrm{i}\sin\theta)]^n \\ &= \sec^n\theta\,(\cos\theta + \mathrm{i}\sin\theta)^n\end{aligned}$$

By de Moivre's theorem,

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$$

So

$$C + \mathrm{i}S = \sec^n \theta(\cos n\theta + \mathrm{i}\sin n\theta)$$

Equating real and imaginary parts gives

$$C = \sec^n \theta \cos n\theta$$

and

$$S = \sec^n \theta \sin n\theta$$

Hence the corresponding expression for S is

$$S = \sec^n \theta \sin n\theta$$

6. The convergent series C and S are defined by

$$C = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n \cos\big((4n+1)\theta\big)$$

$$S = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n \sin\big((4n+1)\theta\big)$$

(a) Show that

$$C + \mathrm{i}S = \frac{2e^{\mathrm{i}\theta}}{2+e^{4\mathrm{i}\theta}}$$ **[4]**

(b) Hence show that

$$C = \frac{4\cos\theta + 2\cos 3\theta}{5 + 4\cos 4\theta}$$ **[4]**

Solution

(a) Using Euler's formula,

$$\cos x + \mathrm{i}\sin x = e^{\mathrm{i}x}$$

so

$$\begin{aligned} C + \mathrm{i}S &= \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n \Big(\cos\big((4n+1)\theta\big) + \mathrm{i}\sin\big((4n+1)\theta\big)\Big) \\ &= \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n e^{\mathrm{i}(4n+1)\theta} \\ &= e^{\mathrm{i}\theta} \sum_{n=0}^{\infty} \left(-\frac{e^{4\mathrm{i}\theta}}{2}\right)^n \end{aligned}$$

This is a geometric series with common ratio

$$r = -\frac{e^{4\mathrm{i}\theta}}{2}$$

and

$$|r| = \frac{1}{2} < 1$$

so it converges and its sum is

$$\begin{aligned} C + \mathrm{i}S &= e^{\mathrm{i}\theta} \cdot \frac{1}{1-r} \\ &= e^{\mathrm{i}\theta} \cdot \frac{1}{1 + \frac{1}{2}e^{4\mathrm{i}\theta}} \\ &= \frac{2e^{\mathrm{i}\theta}}{2+e^{4\mathrm{i}\theta}} \end{aligned}$$

Hence

$$C + \mathrm{i}S = \frac{2e^{\mathrm{i}\theta}}{2+e^{4\mathrm{i}\theta}}$$

(b) Starting from part (a),

$$C + \mathrm{i}S = \frac{2e^{\mathrm{i}\theta}}{2+e^{4\mathrm{i}\theta}}$$

Multiply top and bottom by $2+e^{-4\mathrm{i}\theta}$:

$$\begin{aligned} C+\mathrm{i}S &= \frac{2e^{\mathrm{i}\theta}(2+e^{-4\mathrm{i}\theta})}{(2+e^{4\mathrm{i}\theta})(2+e^{-4\mathrm{i}\theta})} \\ &= \frac{4e^{\mathrm{i}\theta}+2e^{-3\mathrm{i}\theta}}{(2+e^{4\mathrm{i}\theta})(2+e^{-4\mathrm{i}\theta})} \end{aligned}$$

Now simplify the denominator:

$$\begin{aligned} (2+e^{4\mathrm{i}\theta})(2+e^{-4\mathrm{i}\theta}) &= 4+2e^{4\mathrm{i}\theta}+2e^{-4\mathrm{i}\theta}+1 \\ &= 5+2\left(e^{4\mathrm{i}\theta}+e^{-4\mathrm{i}\theta}\right) \\ &= 5+4\cos 4\theta \end{aligned}$$

since $e^{\mathrm{i}x}+e^{-\mathrm{i}x}=2\cos x$.

Therefore

$$C+\mathrm{i}S = \frac{4e^{\mathrm{i}\theta}+2e^{-3\mathrm{i}\theta}}{5+4\cos 4\theta}$$

The denominator is real, so taking real parts gives

$$\begin{aligned} C &= \frac{4\cos\theta+2\cos(-3\theta)}{5+4\cos 4\theta} \\ &= \frac{4\cos\theta+2\cos 3\theta}{5+4\cos 4\theta} \end{aligned}$$

Hence

$$C = \frac{4\cos\theta+2\cos 3\theta}{5+4\cos 4\theta}$$

7. Let

$$C = \sum_{r=0}^{n-1} \cos^2\left(\theta + \frac{2\pi r}{n}\right)$$

$$S = \sum_{r=0}^{n-1} \sin\left(\theta + \frac{2\pi r}{n}\right)\cos\left(\theta + \frac{2\pi r}{n}\right)$$

where n is an integer greater than 2. Let $\exp(x)$ denote e^x.

By considering the real and imaginary parts of

$$\sum_{r=0}^{n-1} \exp\left(2\mathrm{i}\left(\theta + \frac{2\pi r}{n}\right)\right)$$

show that $C = \dfrac{n}{2}$ and $S = 0$. **[7]**

Solution

Let

$$Z = \sum_{r=0}^{n-1} \exp\left(2\mathrm{i}\left(\theta + \frac{2\pi r}{n}\right)\right)$$

Using $\exp(\mathrm{i}x) = \cos x + \mathrm{i}\sin x$, this becomes

$$Z = \sum_{r=0}^{n-1}\left(\cos\left(2\theta + \frac{4\pi r}{n}\right) + \mathrm{i}\sin\left(2\theta + \frac{4\pi r}{n}\right)\right)$$

So

$$\begin{aligned}\mathrm{Re}(Z) &= \sum_{r=0}^{n-1}\cos\left(2\theta + \frac{4\pi r}{n}\right)\\ \mathrm{Im}(Z) &= \sum_{r=0}^{n-1}\sin\left(2\theta + \frac{4\pi r}{n}\right)\end{aligned}$$

Now rewrite Z as a geometric series:

$$\begin{aligned}Z &= \sum_{r=0}^{n-1}\exp\left(2\mathrm{i}\theta + \frac{4\pi \mathrm{i} r}{n}\right)\\ &= \exp(2\mathrm{i}\theta)\sum_{r=0}^{n-1}\left(\exp\left(\frac{4\pi\mathrm{i}}{n}\right)\right)^r\end{aligned}$$

Let

$$\omega = \exp\left(\frac{4\pi\mathrm{i}}{n}\right)$$

Then

$$Z = \exp(2\mathrm{i}\theta)\sum_{r=0}^{n-1}\omega^r$$

Since $n > 2$, we have $0 < \frac{2}{n} < 1$, so $\omega \neq 1$. Therefore the geometric series formula gives

$$Z = \exp(2\mathrm{i}\theta)\frac{1-\omega^n}{1-\omega}$$

But

$$\begin{aligned}\omega^n &= \left(\exp\left(\frac{4\pi\mathrm{i}}{n}\right)\right)^n\\ &= \exp(4\pi\mathrm{i})\\ &= 1\end{aligned}$$

So

$$\begin{aligned} Z &= \exp(2\mathrm{i}\theta)\frac{1-1}{1-\omega} \\ &= 0 \end{aligned}$$

Hence both the real and imaginary parts of Z are zero:

$$\sum_{r=0}^{n-1} \cos\left(2\theta + \frac{4\pi r}{n}\right) = 0$$
$$\sum_{r=0}^{n-1} \sin\left(2\theta + \frac{4\pi r}{n}\right) = 0$$

Now use the double-angle identities.
For C,

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

with $x = \theta + \frac{2\pi r}{n}$. Then

$$\begin{aligned} C &= \sum_{r=0}^{n-1} \cos^2\left(\theta + \frac{2\pi r}{n}\right) \\ &= \sum_{r=0}^{n-1} \frac{1+\cos\left(2\theta + \frac{4\pi r}{n}\right)}{2} \\ &= \frac{1}{2}\sum_{r=0}^{n-1} 1 + \frac{1}{2}\sum_{r=0}^{n-1} \cos\left(2\theta + \frac{4\pi r}{n}\right) \\ &= \frac{1}{2}(n) + \frac{1}{2}(0) = \frac{n}{2} \end{aligned}$$

For S,

$$\sin x \cos x = \frac{1}{2}\sin 2x$$

so, again with $x = \theta + \frac{2\pi r}{n}$,

$$\begin{aligned} S &= \sum_{r=0}^{n-1} \sin\left(\theta + \frac{2\pi r}{n}\right)\cos\left(\theta + \frac{2\pi r}{n}\right) \\ &= \frac{1}{2}\sum_{r=0}^{n-1} \sin\left(2\theta + \frac{4\pi r}{n}\right) \\ &= \frac{1}{2}(0) = 0 \end{aligned}$$

Therefore

$$C = \frac{n}{2} \qquad \text{and} \qquad S = 0$$