

CORE PURE / COMPLEX NUMBERS

Complex Numbers Basics

Worked solutions

8 questions · 62 marks

Questions

Question 1	7 marks	2
Question 2	5 marks	3
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Question 8	9 marks	12

1. (a) Let $z = x + iy$, where $x, y \in \mathbb{R}$.

(i) Write down z^* , the complex conjugate of z , in terms of x and y . [1]

(ii) Hence show that the product of a complex number and its conjugate is always real. [2]

(b) The complex number w satisfies

$$2w + 12 = 5w^* + 14i$$

(i) Find w , giving your answer in the form $a + bi$, where $a, b \in \mathbb{R}$. [3]

(ii) Calculate $w^2(w^*)^2$. [1]

Solution

(a) (i) The complex conjugate is:

$$z^* = x - iy$$

(ii) Using part (a)(i) and $i^2 = -1$:

$$\begin{aligned} zz^* &= (x + iy)(x - iy) \\ &= x^2 - ixy + ixy - i^2y^2 \\ &= x^2 + y^2 \in \mathbb{R} \end{aligned}$$

(b) (i) Taking the complex conjugate of the given equation gives $2w^* + 12 = 5w - 14i$. Rearranging both equations:

$$\begin{aligned} 2w - 5w^* &= -12 + 14i \\ -5w + 2w^* &= -12 - 14i \end{aligned}$$

Multiplying the first equation by 2 and the second by 5, then adding, eliminates w^* :

$$\begin{aligned} -21w &= -84 - 42i \\ w &= 4 + 2i \quad (a = 4, b = 2) \end{aligned}$$

(ii) Since $w^* = 4 - 2i$:

$$\begin{aligned} w^2(w^*)^2 &= (4 + 2i)^2(4 - 2i)^2 \\ &= (12 + 16i)(12 - 16i) \\ &= 12^2 + 16^2 = 400 \end{aligned}$$

2. The complex number z satisfies

$$4(z + 2i) = (1 - 2i)z^*$$

where z^* denotes the complex conjugate of z .

Determine z , giving your answer in the form $a + bi$, where a and b are real.

[5]

Solution

Let $z = a + bi$, where $a, b \in \mathbb{R}$. Then $z^* = a - bi$, so the given equation becomes

$$4(a + bi + 2i) = (1 - 2i)(a - bi)$$

$$4a + (4b + 8)i = a - bi - 2ai + 2bi^2$$

$$4a + (4b + 8)i = (a - 2b) + (-2a - b)i$$

Equating real and imaginary parts gives

$$4a = a - 2b \quad \implies \quad 3a + 2b = 0$$

$$4b + 8 = -2a - b \quad \implies \quad 2a + 5b = -8$$

From the first equation, $a = -\frac{2b}{3}$. Substituting into the second:

$$2\left(-\frac{2b}{3}\right) + 5b = -8$$

$$\frac{11b}{3} = -8$$

$$b = -\frac{24}{11}$$

$$a = -\frac{2}{3}\left(-\frac{24}{11}\right) = \frac{16}{11}$$

Thus, in the required form $a + bi$,

$$z = \frac{16}{11} - \frac{24}{11}i$$

3. The complex numbers z_1 and z_2 are given by

$$z_1 = \sqrt{3} \left(\cos \left(\frac{\pi}{6} \right) + i \sin \left(\frac{\pi}{6} \right) \right)$$

and

$$z_2 = 2 \left(\cos \left(\frac{5\pi}{6} \right) - i \sin \left(\frac{5\pi}{6} \right) \right)$$

(a) State the exact value of

- (i) $|z_1 z_2|$,
- (ii) the principal argument of $z_1 z_2$, taking $-\pi < \arg(z_1 z_2) \leq \pi$.

[2]

Let $w = z_1 z_2$.

(b) Determine

- (i) the smallest positive integer n for which w^n is a positive real number,
- (ii) the exact value of $|w^n|$ for this value of n .

[3]

Solution

(a) (i) The modulus of a product is the product of the moduli:

$$|z_1 z_2| = |z_1| |z_2| = \sqrt{3} \times 2 = 2\sqrt{3}$$

(ii) The minus sign before $i \sin\left(\frac{5\pi}{6}\right)$ means that z_2 has argument $-\frac{5\pi}{6}$. Adding the arguments gives a value in the required principal range:

$$\arg(z_1 z_2) = \frac{\pi}{6} - \frac{5\pi}{6} = -\frac{2\pi}{3}$$

(b) (i) By de Moivre's theorem, raising w to the power n multiplies its argument by n . For w^n to be positive real, this angle must be a whole multiple of 2π , so:

$$\frac{n\left(-\frac{2\pi}{3}\right)}{2\pi} = -\frac{n}{3} \in \mathbb{Z}$$

Thus n must be divisible by 3, and the smallest positive value is:

$$n = 3$$

(ii) Using $|w| = 2\sqrt{3}$ from part (a)(i) and $n = 3$ from part (b)(i):

$$|w^3| = |w|^3 = (2\sqrt{3})^3 = 24\sqrt{3}$$

4. In this question the principal argument of a complex number lies in the interval $[0, 2\pi)$. Complex numbers z_1 and z_2 are defined by $z_1 = 4 + 3i$ and $z_2 = -12 + 5i$.

- (a) Determine $z_1 z_2$, giving your answer in the form $a + bi$. [2]
- (b) Express z_2 in modulus-argument form. [2]
- (c) Verify, by direct calculation, that $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$. [3]

Solution

- (a) Expanding and using $i^2 = -1$:

$$\begin{aligned} z_1 z_2 &= (4 + 3i)(-12 + 5i) \\ &= -48 + 20i - 36i + 15i^2 \\ &= -63 - 16i \quad (a = -63, b = -16) \end{aligned}$$

- (b) The modulus is:

$$|z_2| = \sqrt{(-12)^2 + 5^2} = 13$$

The point $(-12, 5)$ lies in the second quadrant, so the principal argument is:

$$\theta = \pi - \arctan\left(\frac{5}{12}\right)$$

Thus the modulus-argument form is:

$$z_2 = 13(\cos \theta + i \sin \theta), \quad \theta = \pi - \arctan\left(\frac{5}{12}\right)$$

- (c) Let $\alpha = \arctan(\frac{3}{4})$ and $\beta = \arctan(\frac{5}{12})$. Using the quadrants of the three complex numbers:

$$\arg(z_1) = \alpha$$

$$\arg(z_2) = \pi - \beta$$

$$\arg(z_1 z_2) = \pi + \arctan\left(\frac{16}{63}\right)$$

The tangent subtraction formula gives:

$$\tan(\alpha - \beta) = \frac{\frac{3}{4} - \frac{5}{12}}{1 + \frac{3}{4} \cdot \frac{5}{12}} = \frac{16}{63}$$

Since $0 < \alpha - \beta < \frac{\pi}{2}$, we have $\alpha - \beta = \arctan(\frac{16}{63})$. Therefore:

$$\begin{aligned} \arg(z_1) + \arg(z_2) &= \alpha + \pi - \beta \\ &= \pi + \arctan\left(\frac{16}{63}\right) \\ &= \arg(z_1 z_2) \end{aligned}$$

5. The complex numbers z_1 and z_2 satisfy

$$\begin{aligned} z_1 &= -3 + 2i \\ |z_1 z_2| &= 26\sqrt{2} \\ \arg(z_1 z_2) &= -\frac{\pi}{4} \end{aligned}$$

- (a) Express z_1 in the form $r(\cos \theta + i \sin \theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$.
Give r exactly and θ in radians to 4 significant figures.

[2]

- (b) Determine z_2 , giving your answer in the form $a + ib$, where a and b are integers.

[6]

Solution

- (a) The real part is negative and the imaginary part is positive, so the argument lies in the second quadrant. Hence

$$\begin{aligned} r &= |z_1| = \sqrt{(-3)^2 + 2^2} = \sqrt{13} \\ \theta &= \pi - \arctan\left(\frac{2}{3}\right) \end{aligned}$$

The required form is

$$z_1 = \sqrt{13}(\cos \theta + i \sin \theta), \quad r = \sqrt{13}, \quad \theta \approx 2.554 \text{ radians (4 s.f.)}$$

- (b) Write $z_2 = a + ib$, where a, b are real. Multiplying gives

$$\begin{aligned} z_1 z_2 &= (-3 + 2i)(a + ib) \\ &= -3a - 3ib + 2ia + 2i^2b \\ &= (-3a - 2b) + i(2a - 3b) \end{aligned}$$

Since the argument is $-\frac{\pi}{4}$, the real part is positive and the imaginary part is the negative of the real part. Thus

$$\begin{aligned} 2a - 3b &= -(-3a - 2b) \\ 2a - 3b &= 3a + 2b \\ a &= -5b \end{aligned}$$

Substituting into the product gives

$$z_1 z_2 = 13b - 13bi$$

Its real part is positive, so $b > 0$. Using the given modulus,

$$\begin{aligned} \sqrt{(13b)^2 + (-13b)^2} &= 26\sqrt{2} \\ 13\sqrt{2}b &= 26\sqrt{2} \\ b &= 2 \\ a &= -5(2) = -10 \end{aligned}$$

Therefore, in the required form,

$$z_2 = -10 + 2i, \quad a = -10, \quad b = 2$$

6. The complex number w is given by

$$w = -6\sqrt{3} - 6i$$

(a) (i) Find $|w|$. [2]

(ii) Find the principal argument of w , taking $-\pi < \arg(w) \leq \pi$. [2]

The complex numbers z_1 and z_2 are given by

$$z_1 = a - 3i, \quad z_2 = 2(\cos \theta + i \sin \theta)$$

where a is a positive real constant and $-\pi < \theta \leq \pi$.

(b) Given that $z_1 z_2 = w$,

(i) find the exact value of a . [3]

(ii) find θ , giving your answer as an exact multiple of π . [3]

Solution

(a) (i) Using the real and imaginary parts of w :

$$|w| = 6\sqrt{(\sqrt{3})^2 + 1^2} = 12$$

(ii) Let ϕ be the principal argument of w . Since $|w| = 12$,

$$\cos \phi = \frac{-6\sqrt{3}}{12} = -\frac{\sqrt{3}}{2}, \quad \sin \phi = \frac{-6}{12} = -\frac{1}{2}$$

The angle satisfying both conditions in $-\pi < \phi \leq \pi$ is

$$\arg(w) = -\frac{5\pi}{6}$$

(b) (i) Expanding the product gives

$$z_1 z_2 = 2[(a \cos \theta + 3 \sin \theta) + i(a \sin \theta - 3 \cos \theta)]$$

Equating real and imaginary parts with w , then dividing by 2:

$$a \cos \theta + 3 \sin \theta = -3\sqrt{3}, \quad a \sin \theta - 3 \cos \theta = -3$$

Squaring and adding eliminates the cross terms:

$$(a \cos \theta + 3 \sin \theta)^2 + (a \sin \theta - 3 \cos \theta)^2 = 27 + 9$$

$$(a^2 + 9)(\cos^2 \theta + \sin^2 \theta) = 36$$

$$a^2 + 9 = 36$$

$$a^2 = 27$$

Since a is positive,

$$a = 3\sqrt{3}$$

- (ii) Using $a = 3\sqrt{3}$ from part (b)(i), divide by z_1 and multiply numerator and denominator by its conjugate:

$$\begin{aligned} z_2 &= \frac{w}{z_1} = \frac{-6\sqrt{3} - 6i}{3\sqrt{3} - 3i} \\ &= \frac{(-6\sqrt{3} - 6i)(3\sqrt{3} + 3i)}{(3\sqrt{3})^2 + 3^2} \\ &= -1 - \sqrt{3}i \end{aligned}$$

Comparing with $z_2 = 2(\cos \theta + i \sin \theta)$ gives

$$\cos \theta = -\frac{1}{2}, \quad \sin \theta = -\frac{\sqrt{3}}{2}$$

The angle satisfying both conditions in $-\pi < \theta \leq \pi$ is

$$\theta = -\frac{2\pi}{3}$$

7. The complex numbers z and w are given by $z = 3 + i$ and $w = 2 + 4i$.

- (a) (i) Find the modulus and principal argument of zw .
 (ii) Express zw in the form $r(\cos \theta + i \sin \theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$.

[4]

- (b) The complex number v satisfies

$$\frac{1}{v} = \frac{1}{w} - \frac{1}{z}$$

Find v in the form $a + ib$, where a and b are real.

[5]

- (c) The complex conjugate of v is denoted by v^* .
 Show that $vv^* = k$, where k is a real number whose value is to be determined.

[2]

Solution

- (a) (i) Multiplying and using $i^2 = -1$:

$$\begin{aligned} zw &= (3 + i)(2 + 4i) \\ &= 6 + 12i + 2i + 4i^2 \\ &= 2 + 14i \end{aligned}$$

Hence the modulus is:

$$|zw| = \sqrt{2^2 + 14^2} = \sqrt{200} = 10\sqrt{2}$$

Both the real and imaginary parts are positive, so zw lies in the first quadrant. Its principal argument is therefore:

$$\arg(zw) = \arctan\left(\frac{14}{2}\right) = \arctan 7 \approx 1.43 \text{ radians}$$

- (ii) Using part (a)(i), $r = 10\sqrt{2}$ and $\theta = \arctan 7$, which lies in the required interval:

$$zw = 10\sqrt{2}(\cos(\arctan 7) + i \sin(\arctan 7))$$

- (b) Combining the fractions:

$$\begin{aligned} \frac{1}{v} &= \frac{1}{w} - \frac{1}{z} \\ &= \frac{z - w}{zw} \\ &= \frac{(3 + i) - (2 + 4i)}{2 + 14i} \\ &= \frac{1 - 3i}{2 + 14i} \end{aligned}$$

Taking reciprocals and multiplying numerator and denominator by the conjugate of $1 - 3i$:

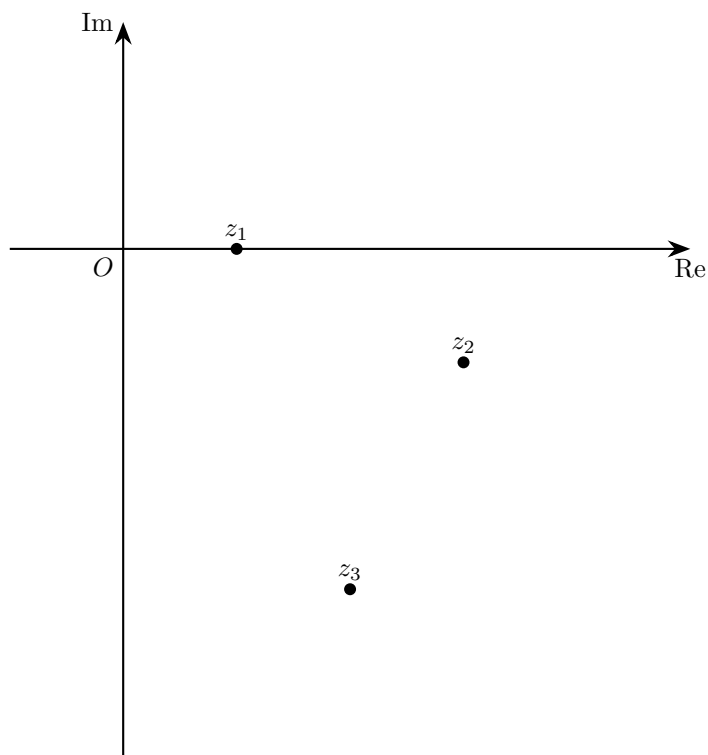
$$\begin{aligned}v &= \frac{2 + 14i}{1 - 3i} \\&= \frac{(2 + 14i)(1 + 3i)}{(1 - 3i)(1 + 3i)} \\&= \frac{2 + 6i + 14i + 42i^2}{1 - 9i^2} \\&= \frac{-40 + 20i}{10} \\&= -4 + 2i\end{aligned}$$

This is of the form $a + ib$, with $a = -4$ and $b = 2$.

(c) The conjugate is $v^* = -4 - 2i$, so:

$$\begin{aligned}vv^* &= (-4 + 2i)(-4 - 2i) \\&= 16 - (2i)^2 \\&= 16 + 4 = 20\end{aligned}$$

Thus $vv^* = k$, where $k = 20 \in \mathbb{R}$.



8. The Argand diagram above shows the complex numbers $z_1 = 1$, $z_2 = 3 - i$ and $z_3 = 2 - 3i$.

(a) Explain why no polynomial equation of degree four with real coefficients can have all three of these numbers as roots. [2]

(b) Show that

$$\arg \left(\frac{z_2 - z_1}{z_3 - z_1} \right) = \frac{\pi}{4} \quad [3]$$

(c) Hence show that

$$\arctan(3) - \arctan\left(\frac{1}{2}\right) = \frac{\pi}{4} \quad [2]$$

(d) Shade, on a copy of the diagram, the region consisting of all points whose complex coordinates satisfy

$$|z - 1| \leq |z - 2 + 3i| \quad [2]$$

Solution

(a) Non-real roots of a polynomial with real coefficients occur in complex conjugate pairs. Thus $3 + i$ and $2 + 3i$ would also have to be roots.

This gives five distinct roots, 1 , $3 - i$, $3 + i$, $2 - 3i$ and $2 + 3i$, whereas a polynomial of degree four has at most four distinct roots.

(b) Substituting the given complex numbers and multiplying by the conjugate of the denominator:

$$\begin{aligned}\frac{z_2 - z_1}{z_3 - z_1} &= \frac{2 - i}{1 - 3i} \\ &= \frac{(2 - i)(1 + 3i)}{(1 - 3i)(1 + 3i)} \\ &= \frac{2 + 6i - i - 3i^2}{1 + 9} \\ &= \frac{5 + 5i}{10} = \frac{1}{2} + \frac{1}{2}i\end{aligned}$$

The real and imaginary parts are equal and positive, so the point lies in the first quadrant at angle $\frac{\pi}{4}$:

$$\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \frac{\pi}{4}$$

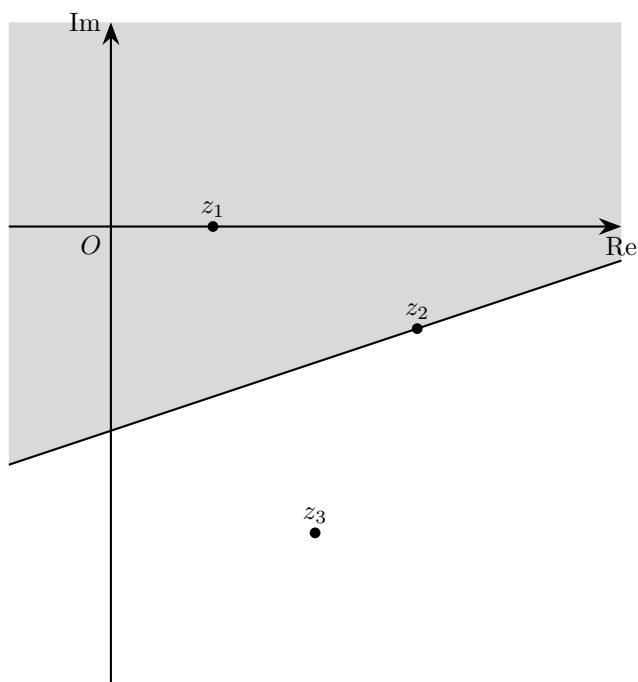
(c) Both $z_2 - z_1 = 2 - i$ and $z_3 - z_1 = 1 - 3i$ lie in the fourth quadrant, so

$$\arg(z_2 - z_1) = -\arctan\left(\frac{1}{2}\right), \quad \arg(z_3 - z_1) = -\arctan(3)$$

Dividing complex numbers subtracts their arguments. Hence, using part (b):

$$\begin{aligned}\frac{\pi}{4} &= \arg(z_2 - z_1) - \arg(z_3 - z_1) \\ &= \arctan(3) - \arctan\left(\frac{1}{2}\right)\end{aligned}$$

(d) The shaded diagram should look like:



The solid boundary is included. Writing $z = x + iy$, and squaring both non-negative moduli, gives:

$$|z - 1| \leq |z - 2 + 3i| \iff (x - 1)^2 + y^2 \leq (x - 2)^2 + (y + 3)^2$$

$$\iff x^2 - 2x + 1 + y^2 \leq x^2 - 4x + 4 + y^2 + 6y + 9$$

$$\iff 2x - 6y \leq 12$$

$$\iff y \geq \frac{x - 6}{3}$$