Questions

1. A complex number z has modulus 1 and argument θ.

(a) Show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta \quad \text{and} \quad z^n - \frac{1}{z^n} = 2\mathrm{i}\sin n\theta, \qquad n \in \mathbb{Z}^+$$

[**3**]

(b) Hence, by expanding $\left(z + \frac{1}{z}\right)^3$ and $\left(z - \frac{1}{z}\right)^3$, show that

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

[**4**]

2. (a) Using de Moivre's theorem, show that

$$\sin^4\theta\cos^2\theta \equiv \frac{1}{32}(\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2)$$

[5]

(b) By using the substitution $\alpha = \left(\frac{\pi}{2} - \theta\right)$, or otherwise, find a similar identity for $\sin^2\theta\cos^4\theta$. **[3]**

(c) Given that $0 < a < \frac{\pi}{2}$ and

$$\int_0^a \left(\sin^4\theta\cos^2\theta + 2\sin^2\theta\cos^4\theta\right)\,\mathrm{d}\theta = \frac{\pi}{32} - \frac{\sqrt{3}}{64}$$

find the exact value of a. **[5]**

3. Use de Moivre's theorem to find the constants A, B and C in the identity

$$\sin\theta\cos^4\theta = A\sin\theta + B\sin 3\theta + C\sin 5\theta$$ **[4]**

4. **(i)** Starting with the result

$$e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$$

show that

(A)

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$$ **[2]**

(B)

$$\sin\theta = \frac{1}{2\mathrm{i}}\left(e^{\mathrm{i}\theta} - e^{-\mathrm{i}\theta}\right)$$ **[2]**

(ii) Using the result in part (i)(A), obtain the values of the constants a, b, c, d and e in the identity

$$\cos 8\theta \equiv a\sin^8\theta + b\sin^6\theta + c\sin^4\theta + d\sin^2\theta + e$$ **[6]**

(iii) Using the result in part (i)(B), obtain the values of the constants P, Q, R, S and T in the identity

$$\sin^8\theta \equiv P\cos 8\theta + Q\cos 6\theta + R\cos 4\theta + S\cos 2\theta + T$$ **[6]**

(iv) Hence show that

$$\sin\frac{\pi}{8} = \left(\frac{17 - 12\sqrt{2}}{64}\right)^{\frac{1}{8}}$$ **[3]**

5. **(a)** Use de Moivre's theorem to show that $\cos 6\theta \equiv (1 - 2\sin^2\theta)(16\sin^4\theta - 16\sin^2\theta + 1)$ **[5]**

(b) By solving the equation $\cos 6\theta = 0$, deduce that $\sin^2\left(\frac{\pi}{12}\right) = \frac{2-\sqrt{3}}{4}$. **[4]**

(c) Hence, or otherwise, write down the exact values of $\sin^2\left(\frac{5\pi}{12}\right)$, $\sin^2\left(\frac{7\pi}{12}\right)$ and $\sin^2\left(\frac{11\pi}{12}\right)$. **[3]**

6. Let $t = \tan\theta$. By writing

$$\cos\theta + \mathrm{i}\sin\theta = \cos\theta\,(1 + \mathrm{i}t)$$

and expanding $(1 + \mathrm{i}t)^6$, use de Moivre's theorem to show that

$$\tan 6\theta = \frac{6\tan\theta - 20\tan^3\theta + 6\tan^5\theta}{1 - 15\tan^2\theta + 15\tan^4\theta - \tan^6\theta}$$ **[7]**

7. **(a)** Use de Moivre's theorem to show that

$$\sin 6x = \sin 2x \left(a\cos^4 x + b\cos^2 x + c\right)$$

where a, b and c are integers to be determined. **[4]**

(b) Hence solve, for $0 < \theta < \frac{\pi}{2}$,

$$\sin 6\theta = \sin 2\theta \cos 2\theta$$

giving your answers to 3 decimal places. **[4]**

8. **(a)** By considering the binomial expansion of $(z - z^{-1})^6$, where $z = \cos\theta + \mathrm{i}\sin\theta$, use de Moivre's theorem to show that

$$\sin^6\theta = \frac{1}{32}\big(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta\big)$$ **[5]**

(b) Hence, using the substitution $x = \tan\theta$, find the exact value of

$$\int_0^1 \frac{x^6}{(1+x^2)^4}\,\mathrm{d}x$$ **[4]**

9. (a) Use de Moivre's theorem to show that

$$2\cos 5\theta = 32\cos^5\theta - 40\cos^3\theta + 10\cos\theta$$ **[4]**

(b) Hence solve the equation

$$x^5 - 5x^3 + 5x + 1 = 0$$

giving all roots in the form $2\cos q\pi$, where q is a rational number. **[4]**

10. **(a)** If n is a positive integer and $z = \cos\theta + \mathrm{i}\sin\theta$, show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta, \qquad z^n - \frac{1}{z^n} = 2\mathrm{i}\sin n\theta$$ **[2]**

(b) By considering $\left(z + \frac{1}{z}\right)^3 \left(z - \frac{1}{z}\right)^2$, find constants A, B and C such that

$$\sin^2\theta\cos^3\theta = A\cos 5\theta + B\cos 3\theta + C\cos\theta$$ **[6]**

11. **(a)** By writing $1+\mathrm{i}\tan\theta = \sec\theta(\cos\theta + \mathrm{i}\sin\theta)$ and using de Moivre's theorem, show that

$$(1+\mathrm{i}\tan\theta)^4 = \frac{\cos 4\theta + \mathrm{i}\sin 4\theta}{\cos^4\theta}$$

[3]

(b) Expand $(1+\mathrm{i}\tan\theta)^4$ and hence show that

$$\cos 4\theta = \frac{1 - 6\tan^2\theta + \tan^4\theta}{(1+\tan^2\theta)^2}$$

and

$$\sin 4\theta = \frac{4\tan\theta(1-\tan^2\theta)}{(1+\tan^2\theta)^2}$$

[4]

(c) Deduce that

$$\tan 4\theta = \frac{4\tan\theta(1-\tan^2\theta)}{1 - 6\tan^2\theta + \tan^4\theta}$$

[2]

12. **(a)** Use de Moivre's theorem to show that

$$\sec 6\theta = \frac{\sec^6 \theta}{32 - 48\sec^2 \theta + 18\sec^4 \theta - \sec^6 \theta}$$ **[6]**

(b) Hence obtain the roots of the equation

$$x^6 - 18x^4 + 48x^2 - 32 = 0$$

in the form $\sec(q\pi)$, where q is rational. **[4]**

13. **(a)** Use de Moivre's theorem to show that

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$ **[5]**

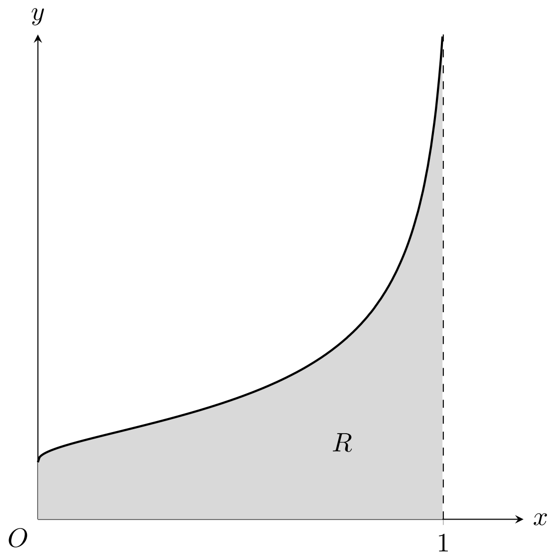
(b) Hence solve for $0 \leq \theta \leq \pi$

$$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = \cos 2\theta$$

giving your answers as exact multiples of π. **[5]**

14. **(a)** Use de Moivre's theorem to determine constants A, B and C such that

$$\cos^5\theta \equiv A\cos 5\theta + B\cos 3\theta + C\cos\theta$$ **[5]**



The function f is defined by

$$f(x) = -3\sin\left(5\cos^{-1}\left(x^{\frac{1}{6}}\right)\right) - 25\sin\left(3\cos^{-1}\left(x^{\frac{1}{6}}\right)\right) - 150\sin\left(\cos^{-1}\left(x^{\frac{1}{6}}\right)\right), \qquad 0 \le x < 1$$

(b) Show that

$$f'(x) = \frac{40}{\sqrt{1 - x^{\frac{1}{3}}}}$$ **[6]**

The diagram shows the curve with equation

$$y = \frac{1}{\sqrt{1 - x^{\frac{1}{3}}}}$$

for $0 \le x < 1$ and the asymptote $x = 1$. The region R is the unbounded region between the curve, the x-axis, the line $x = 0$ and the line $x = 1$.
You are given that the area of R is finite.

(c) Determine the exact area of R. **[3]**

15. **(i)** Use de Moivre's theorem to show that

$$\cos 5\theta \equiv \cos\theta\left(1 - 12\sin^2\theta + 16\sin^4\theta\right)$$ **[5]**

(ii) Hence determine the range of values of

$$\frac{\cos 5\theta}{\cos\theta}$$

given that $\cos\theta \neq 0$. **[7]**

16. (a) Use de Moivre's theorem to show that

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

[5]

(b) Use the identity given in part (a) to find the 2 positive roots of

$$64x^5 - 80x^3 + 20x + 1 = 0$$

giving your answers to 3 significant figures. **[4]**

17. **(a)** Use de Moivre's theorem to show that

$$\sin^2\theta\cos^2\theta = \frac{1}{8}(1-\cos 4\theta)$$

[5]

(b) Hence, or otherwise, find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} - 2y\tan\theta = \sin^2\theta$$

for which $y = 0$ when $\theta = \frac{1}{4}\pi$. **[6]**

18. **(a)** Use de Moivre's theorem to show that

$$\cos^6\theta = a\cos 6\theta + b\cos 4\theta + c\cos 2\theta + d$$

where a, b, c and d are constants to be found. **[5]**

(b) Hence show that

$$\int_0^{\pi/4} \cos^6\theta \, \mathrm{d}\theta = \frac{11}{48} + \frac{5\pi}{64}$$ **[5]**