1. For each positive integer n, define

$$E_n = n^4 + 4$$

For which one of the following statements is $n = 3$ a counterexample?

A For all positive integers n, E_n is divisible by $n^2 - 2n + 2$

B For all positive integers n, E_n is divisible by $n^2 + 2n + 2$

C For all positive integers n, E_n is **not** divisible by 17

D For all positive integers n, E_n is **not** divisible by 3

E For all positive integers n, E_n is divisible by 5

F For all positive integers n, E_n has remainder 1 when divided by 4

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2. A student is asked to prove the statement:

if u **and** v are real numbers such that $u^2 + v^2 = 0$, **then** $u = 0$ **and** $v = 0$

They write the following argument:

I For all real w, $w^2 \geq 0$

II So $u^2 \geq 0$ **and** $v^2 \geq 0$

III If the sum of two real numbers is 0 and each is **not** negative, then each number is 0

IV Therefore $u^2 = 0$ **and** $v^2 = 0$

V Hence $u = 0$ **and** $v = 0$

Which of the following best describes this argument?

A It is a correct direct proof of the statement.

B It is incorrect, because line I is false.

C It is incorrect, because line III is false.

D It is incorrect, because line IV does not follow from the previous lines.

E It is incorrect, because the reasoning would only work if u **and** v were positive.

F It is a correct proof of the converse, but **not** of the original statement.

3. We say a sequence of real numbers x_n is *a Cauchy sequence* if

for all $\varepsilon > 0$ there exists a positive integer $N \in \mathbb{N}$ such that
for all $m, n \geq N$ we have $|x_m - x_n| < \varepsilon$

Which one of the following statements is true if and only if x_n is **not** a Cauchy sequence?

A There exists $\varepsilon > 0$ such that for every $N \in \mathbb{N}$ there exist $m, n \geq N$ with $|x_m - x_n| \geq \varepsilon$

B For every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ and there exists $m, n \geq N$ with $|x_m - x_n| \geq \varepsilon$

C There exists $N \in \mathbb{N}$ such that for all $\varepsilon > 0$ and all $m, n \geq N$ we have $|x_m - x_n| \geq \varepsilon$

D For every $\varepsilon > 0$ and every $N \in \mathbb{N}$ there exists $m, n \geq N$ with $|x_m - x_n| < \varepsilon$

E There exists $\varepsilon > 0$ and there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$ we have $|x_m - x_n| \geq \varepsilon$

F For every $N \in \mathbb{N}$ there exists $\varepsilon > 0$ such that for all $m, n \geq N$ we have $|x_m - x_n| \geq \varepsilon$

G There exists $\varepsilon > 0$ such that for every $N \in \mathbb{N}$ and for all $m, n \leq N$ we have $|x_m - x_n| \geq \varepsilon$

H There exists $\varepsilon > 0$ such that for every $N \in \mathbb{N}$ there exist $m, n \geq N$ with $|x_m - x_n| < \varepsilon$

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4. Let S be the statement about integers n:

$(*)$ For every integer n there exists an integer m such that $m > n$ **and** m is even

Which of the following statements is **logically equivalent** to the **negation** of $(*)$?

A For some integer n there exists an integer m such that $m > n$ **and** m is odd.

B For some integer n, **if** an integer m is greater than n **then** m is even.

C For every integer n, **if** n is even **then** there is no integer $m > n$ that is even.

D It is **not** the case that there exists an integer m which is greater than every integer n **and** is even.

E For every integer m there exists an integer $n < m$ such that m is odd.

F There exists an integer n such that there is exactly one even integer greater than n.

G There exists an integer n such that for all integers m, **if** $m > n$ **then** m is **not** even.

5. Consider the statement about integers n:

$(*)$ For every integer n, **if** n is even **then** $n^2 + 2n$ is divisible by 4

Which of the following statements has the same **truth value** as $(*)$?

I: For all integers n, **if** $n^2 + 2n$ is **not** divisible by 4 **then** n is odd

II: There exists an integer n which is even **and** for which $n^2 + 2n$ is divisible by 4

III: For some integer n, **if** n is even **then** $n^2 + 2n$ is divisible by 4

A **I** only

B **II** only

C **III** only

D **I** and **II** only

E **I** and **III** only

F **II** and **III** only

G **I**, **II** and **III**

H none of them

6. A statement S is given:

For all real numbers x, **if** $x^2 \geq 1$ **then** $x^4 \geq x^2$

A student wishes to prove S by contradiction and writes the following argument:

Assume that there exists a real number x such that $x^2 \geq 1$ **and** $x^4 < x^2$

I Then $x^4 - x^2 < 0$

II So $x^2(x^2 - 1) < 0$

III So $x^2 - 1 < 0$

IV So $x^2 < 1$, which contradicts $x^2 \geq 1$

Therefore the original statement S is concluded to be true.

Which of the following best describes this argument?

A It is completely correct.

B It is incorrect, but it would be correct if written in the reverse order.

C It is incorrect, and the student has actually shown that S is false.

D It is incorrect because line II does not follow from line I.

E It is incorrect because line III does not follow from line II.

F It is incorrect because the student has **not** really used proof by contradiction.

7. For real numbers a and b, consider the statement $S(a,b)$:

$$\textbf{if } ab < 0 \textbf{ then } a^2 + b^2 > 0$$

Which of the following descriptions is correct?

A $S(a,b)$ is false for some real a, b because $a^2 + b^2$ can be negative

B $S(a,b)$ is false for some real a, b because $a^2 + b^2 = 0$ can occur when $ab < 0$

C $S(a,b)$ is false for some real a, b because $ab < 0$ forces $a^2 + b^2 = 0$

D $S(a,b)$ is true for all real a, b, and the condition $ab < 0$ is **necessary** for $a^2 + b^2 > 0$

E $S(a,b)$ is true for all real a, b, and the condition $ab < 0$ is **sufficient and necessary** for $a^2 + b^2 > 0$

F $S(a,b)$ is true for all real a, b, and the condition $ab < 0$ is **sufficient** but **not necessary** for $a^2 + b^2 > 0$

G $S(a,b)$ is true for all real a, b, and the condition $ab < 0$ is neither **necessary** nor **sufficient** for $a^2 + b^2 > 0$

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8. Consider the statement:

For every positive integer n there exists a positive integer k such that for all positive integers m, the number $n(m+k)$ is **not** a square

Which one of the following is the **negation** of this statement?

A For every positive integer n there exists a positive integer k such that for every positive integer m,the number $n(m+k)$ is a square.

B There exists a positive integer n such that for every positive integer k **and** every positive integer m, the number $n(m+k)$ is **not** a square.

C There exists a positive integer n such that there exists a positive integer k for which $n(m+k)$ is a square for all positive integers m.

D For every positive integer n, for every positive integer k, there exists a positive integer m such that $n(m+k)$ is **not** a square.

E There exists a positive integer n such that for every positive integer k there exists a positive integer m such that $n(m+k)$ is a square.

9. Let $f(x) = ax^2 + bx + c$ be a quadratic function with real coefficients such that $f(x) \geq 0$ for all real x

Suppose that for every real x,

$$f(x+1) - f(x) \geq 2x + 3$$

What is the smallest possible value of $f(0)$?

A 0

B $\frac{1}{4}$

C 1

D 2

E $\frac{9}{4}$

F 4

G $\frac{25}{4}$

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10. Let C be the circle

$$x^2 + y^2 = 9$$

Let P be the point $(a, 0)$ where $a > 3$

The two tangents from P to C touch the circle at T_1 and T_2

What is the area of triangle PT_1T_2 in terms of a?

A $3\sqrt{a^2 - 9}$

B $\frac{3(a^2 - 9)^{3/2}}{a^2}$

C $\frac{9}{a}\sqrt{a^2 - 9}$

D $\frac{a}{3}\sqrt{a^2 - 9}$

E $\frac{a^2 - 9}{3}$

F $\frac{a^2 - 9}{a}$

G $\frac{9}{\sqrt{a^2 - 9}}$

11. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x

Define

$$f(x) = \lfloor 2x \rfloor - \lfloor x \rfloor$$

for all real x

The value of $\int_0^2 f(x)\,\mathrm{d}x$ is

A 0

B $\frac{1}{2}$

C 1

D $\frac{3}{2}$

E 2

F 3

G 4

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12. A sequence (a_n) is defined by $a_1 = 1$ and, for $n \geq 1$

$$a_{n+1} = \frac{a_n + 4}{\sqrt{a_n + 1}}$$

Which of the following statements is true?

A (a_n) is strictly decreasing for all n

B (a_n) is bounded above but **not** bounded below

C (a_n) is bounded below by 1 and bounded above by 4

D (a_n) is bounded below by 2 and bounded above by 4

E (a_n) is bounded below by 1 but **not** bounded above

F $a_n > 4$ for all $n \geq 2$

G The sequence (a_n) converges to 4

13. Let P and Q be statements

Suppose exactly one of (**if** P **then** Q) and (**if** Q **then** P) is true

Suppose further that (P **or** Q) is true

Which statement must be true?

A (P **and** Q) is true

B Exactly one of P and Q is true

C Both P and Q are false

D (**if** $\neg P$ **then** $\neg Q$) is true

E (P **if and only if** Q) is true

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14. Let x be a real number

Which one of the following statements is a **sufficient** condition for exactly two of the other four statements?

A $x \geq 4$

B $x > 1$

C $x^2 \geq 10$

D $x \geq (x-4)^2$

E $0 \leq x \leq 3$

15. A non-empty finite set S of whole numbers is called *linked* **if and only if**

for every a in S, there exists b in S with $b \neq a$ such that $\gcd(a,b) > 1$

where $\gcd(a,b)$ denotes the greatest common divisor (also known as the highest common factor) of integers a and b

Which of the following is true **if and only if** S is **not** linked?

A For every a in S, there exists b in S with $b \neq a$ such that $\gcd(a,b) = 1$

B For every a in S, there exists a prime factor of a that divides every element of S

C There exists a in S such that for every b in S with $b \neq a$ we have $\gcd(a,b) > 1$

D There exists a in S such that for every b in S with $b \neq a$ we have $\gcd(a,b) = 1$

E There exists a prime p such that p divides exactly one element of S

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16. Consider the statement:

$$(*) \quad \textbf{if } 0 \leq f(x) \leq 1 \text{ for all real } x \text{ with } 0 \leq x \leq 1, \textbf{ then} \quad \int_0^1 f(x)\,\mathrm{d}x \leq \int_0^1 (f(x))^2\,\mathrm{d}x$$

Which of the following functions is a *counterexample* to $(*)$?

A $f(x) = 2x$

B $f(x) = x^2 + \frac{1}{2}$

C $f(x) = \frac{1}{2}$

D $f(x) = 1 + x$

E $f(x) = 0$

F $f(x) = 5x(1-x)$

G $f(x) = \sqrt{x} + \frac{1}{2}$

H $f(x) = 1$

17. Consider the claim:

For all integers a, b, c, **if** $ab = ac$ **then** $b = c$

A student's argument is as follows:

I Suppose $ab = ac$

II Then $ab - ac = 0$

III So $a(b - c) = 0$

IV Therefore $b - c = 0$

V Hence $b = c$

Which of the following best describes this argument?

A The argument is completely correct.

B The argument is incorrect, and the first error is on line I

C The argument is incorrect, and the first error is on line II

D The argument is incorrect, and the first error is on line III

E The argument is incorrect, and the first error is on line IV

F The argument is incorrect, and the first error is on line V

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18. Let f be a function defined for all real x, and assume the integral exists

Which one of the following is a **sufficient** condition for

$$\int_{-2}^{4} f(x)\,\mathrm{d}x = 0$$

A $f(1) = 0$

B $f(-2) = f(4) = 0$

C $f(-x) = -f(x)$ for all real x

D $f(x + 1) = -f(1 - x)$ for all real x

E $f(x + 2) = -f(2 - x)$ for all real x

F $f(x - 1) = -f(1 - x)$ for all real x

19. Consider the two statements about a real-valued function f:

(A) For all real numbers x there exists a real number y such that $y > x$ **and** $f(y) < f(x)$

(B) There exists a real number y such that for all real numbers x with $x < y$, $f(y) < f(x)$

Which of the following is true?

A Both (A) **and** (B) are always true for any function f.

B For some functions f, (A) is true **and** (B) is true, but **not** for all f.

C There is a function f for which (A) is true **and** (B) is false, but **not** the other way round.

D There is a function f for which (B) is true **and** (A) is false, but **not** the other way round.

E For every function f, (A) **and** (B) are either both true or both false.

F There is no function f for which (A) is true.

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20. The positive real numbers $p \times 10^{-4}$, $q \times 10^{-2}$ and $r \times 10^{-1}$ are each in standard form, and

$$(p \times 10^{-4}) + (q \times 10^{-2}) = (r \times 10^{-1})$$

How many of the following statements must be true?

I: $q > 1$
II: $r > 1$
III: $q < r$
IV: $p < r$

A 1 of them

B 2 of them

C 3 of them

D 4 of them

E None of them